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THE DIELECTRIC PROPERTIES OF SOLUTIONS OF ELECTROLYTES IN A NON-POLAR SOLVENT

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The dielectric properties of solutions of electrolytes in polar solvents have been the subject of numerous investigations, theoretical as well as experimental. According to the current theory, electrolytes in solution are largely, if not completely, dissociated into their constituent ions, especially in solvents of higher dielectric constant. The problem in the case of such solutions, therefore, is to determine the influence of the ions upon the dielectric constant of the medium. Although a given ion is surrounded by a net charge of opposite sign, there is no specific interaction between this ion and any other.

It is well known that many electrolytes are soluble in non-polar solvents, such as benzene. The properties of these solutions indicate that the dissolved electrolytes are not largely dissociated into their ions. Vingee[1] and Batson[2] have studied the freezing point depression of solutions of electrolytes in benzene and dioxane, and Fuoss and Kraus[3] have studied the conductance of solutions of numerous electrolytes in the same solvents. The conductance is of such a low order that a high degree of dissociation of the electrolyte would seem to be excluded. Fuoss and Kraus have advanced, and, in some measure, substantiated the hypothesis that, in solvents of very low dielectric constant, Coulomb forces acting between the ions are sufficiently great to cause the ions to pair up and to render such ion-pairs stable. They have further suggested that, at higher concentrations, the ion-pairs combine to form more complex aggregates. The measurements of Vingee and of Batson, on the depression of the freezing points of solutions of electrolytes in dioxane and benzene, indicate that, while at extremely low concentrations the freezing point depression approaches that required for a solute of molecular weight equal to the formula weight of the electrolyte, at higher concentrations, the freezing point depression is much below this value. These results may be interpreted as indicating that at very low concentrations the electrolyte is present in the form of ion-pairs,

while at higher concentrations these pairs unite to form more complex aggregates. Even though the precise mechanism involved may remain in doubt, there is strong evidence indicating that specific interaction takes place between the ions.

If ion-pairs are present in solution, the charges must be separated by distances of the order of 10^{-8} cm. and the ion-pairs should orient themselves under the action of an impressed field. Solutions of electropolar molecules of this type, therefore, should show a rapidly increasing dielectric constant with increasing concentration.

The only observations recorded in the literature relating to the dielectric constant of solutions of an electrolyte in a non-polar medium are those of Williams and Allgeier,[4] who measured the dielectric constant of solutions of silver perchlorate in benzene between 0.01 and 0.1 N. From their measurements, they concluded that, in the region measured, the molar polarization of the solution varies as a linear function of the concentration of the electrolyte, and deduced the value 477.0 cc. for the limiting molar polarization of silver perchlorate at zero concentration. In the light of the foregoing considerations, this value for the molar polarization of silver perchlorate seemed surprisingly low and led us to suspect that larger values would be found if the measurements were extended to lower concentrations.

We have measured the dielectric constant of solutions of tetraisoamylammonium picrate, tri-isoamylammonium picrate, tetraisoamylammonium bromide and silver perchlorate in benzene at 25°. The detailed results need not be given here, since they will be published elsewhere. The method employed for measuring the dielectric constant of the solutions was substantially that of Wyman[5] and was modified only by introducing a sensitive precision condenser in the circuit of the variable oscillator in order to interpolate between the harmonics of a fixed frequency crystal oscillator. This served to fix the frequency of the variable oscillator and, therefore, that of the resonator containing the solution to be measured. The measurements were carried out at 25° and to concentrations as low as 10^{-4} N.

The results are shown graphically in the accompanying figure, in which the increase of the dielectric constant of the solution over that of the pure solvent is plotted as ordinate against the concentration (in moles per liter) as abscissa. For purposes of comparison, the dielectric constant change for solutions of *meta*dinitrobenzene, a rather highly-polar molecule of ordinary type, is shown on the same figure.

It will be noted that the increase in dielectric constant due to the electrolytes is enormously greater than that due to *meta*dinitrobenzene. At 5×10^{-4} N, the dielectric constant increase due to tetraisoamylammonium picrate, tri-isoamylammonium picrate, silver perchlorate and tetraisoamylammonium bromide is, respectively, 18, 10, 6 and 5.5 times that of *meta*dinitrobenzene. In other words, the magnitude of the effect observed in

the case of electrolytes is of a different order from that of ordinary polar molecules.

All the curves are concave toward the axis of concentration, but, while the curves for the two picrates are very nearly linear, particularly at lower concentrations, those for the bromide and the perchlorate show a high degree of curvature. This is especially true of tetraisoamylammonium bromide, where marked curvature persists down to the lowest concentration measured. At the higher concentrations, our values for silver per-

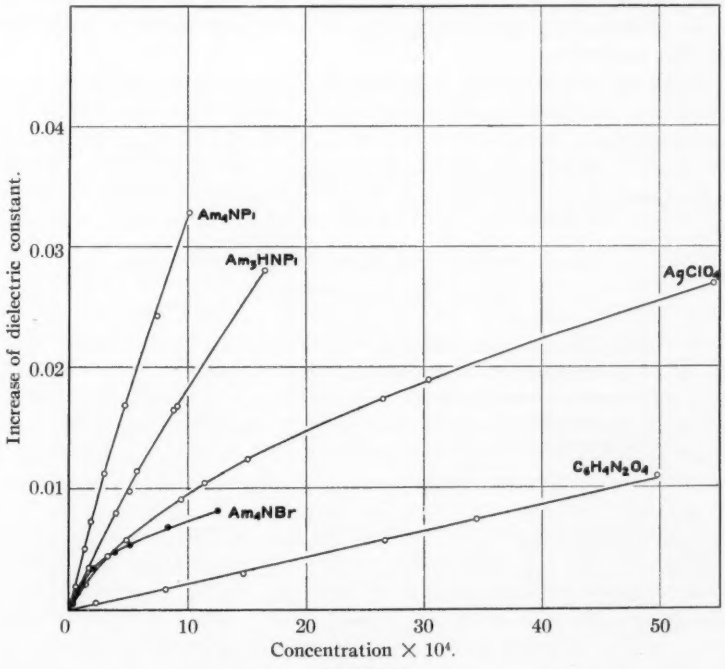

FIGURE 1

Change of dielectric constant of solutions of electrolytes in benzene.

chlorate agree with those of Williams and Allgeier but, as may be seen from the figure, the dielectric constant effect changes greatly between 0.01 *N* and 0.0001 *N*.

In order to determine the molar polarization of the substances, it is necessary to extrapolate either the dielectric constant curves themselves or the molecular polarization of the solutions to zero concentration. As Hedestrand[6] has pointed out, if the dielectric constant curves are non-linear, it is preferable to determine the limiting tangent of the dielectric

constant curve rather than to extrapolate the curve of molecular polarization. The limiting values of the tangents to the different curves have been determined graphically, and, combining with these the linear density effects of the solutions, limiting values have been computed for the molar polarization of the four electrolytes. In general, these values represent lower limits, since the dielectric constant curve is extrapolated approximately linearly from the lowest concentration at which measurements were made to zero concentration. Especially in the case of tetraisoamylammonium bromide, the curve exhibits marked curvature even at the lowest concentrations. Our limiting tangent, therefore, is probably considerably lower than the true tangent.

Values for the molecular polarization are given in the following table. Values are also given for the polar moment of the molecules, calculated in the usual way. The quantity "a" given in the last column was obtained by dividing the value for the polar moment by the unit charge. It is an approximation to the distance between the centers of charge in an ion-pair.

CONSTANTS FOR ELECTROLYTES DISSOLVED IN BENZENE

SUBSTANCE	MOLAR POLARIZATION, CC.	ELECTRIC MOMENT $\times 10^{18}$	"a" $\times 10^{8}$
Tetraisoamylammonium picrate	6725	18.00	3.78
Tetraisoamylammonium bromide	4442	14.7	3.08
Tri-isoamylammonium picrate	3475	12.91	2.71
Silver perchlorate	3020	11.97*	2.56
*Meta*dinitrobenzene	367	4.00*	0.838

* Corrected for distortion polarization.

In calculating the electric moments for the first three salts, the molecular polarization has not been corrected for polarization due to distortion. This correction, even in the case of the most complex molecules, would hardly amount to 200 cc. and would, therefore, come within the limits of error of the determinations. For tetraisoamylammonium picrate, the molecular polarization is approximately ten times that of ordinary, highly-polar molecules. Correspondingly, its polar moment is extraordinarily high. The distance between the centers of ionic charge varies for the four electrolytes from 2.6×10^{-8} to 3.8×10^{-8} cm. These values seem reasonable and agree as well as might be expected with the "a" values calculated by Fuoss and Kraus from conductance measurements.

The approximately linear curves of the two picrates, as contrasted with the rather highly eccentric curve for tetraisoamylammonium bromide, are of interest. The latter salt has two symmetrical ions, while of the former, one has an unsymmetrical anion and the other has a somewhat unsymmetrical cation, in addition. Evidently, the symmetry of its ions has

a marked influence on the dielectric behavior of an electrolyte at higher concentrations. The conductance and the freezing point curves of the same electrolytes exhibit similar differences.

[1] Vingee, Thesis, Brown University, 1931.
[2] Batson, Unpublished observations in This Laboratory.
[3] Fuoss and Kraus, *J. Am. Chem. Soc.*, **55**, 21, 1019, 2387, 3614 (1933).
[4] Williams and Allgeier, *Ibid.*, **49**, 2416 (1927).
[5] Wyman, *Phys. Rev.*, **35**, 623 (1930).
[6] Hedestrand, *Z. phys. Chem.*, **B 22**, 1 (1933).

THE PURIFICATION AND PHYSICAL PROPERTIES OF CHEMICAL COMPOUNDS. IV. A DEVELOPMENT OF A THEORETICAL BASIS FOR THE BEHAVIOR OF CONTROLLED TIME-TEMPERATURE CURVES

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In order to make a study of the agreement attainable between the theoretical time-temperature curve and the experimental data as determined by means of an apparatus already described in the literature,[1] it was necessary to develop the proofs of the two propositions given below.

Let it be assumed (1) that a one-gram sample of a chemical compound hermetically sealed in a container of negligible heat capacity be suspended in a vacuum within a copper shield of high heat capacity; (2) that the sample is at constant pressure and that it is at all times homogeneous and at a uniform temperature throughout; and (3) that the sample is thermally insulated so that all heat transfer to or from the surroundings is through radiation and that the rate of such heat transfer can be expressed by Newton's Law of Cooling,

$$dH = K(\theta_s - \theta)dt \qquad (1)$$

where H is the heat content per gram of the sample at time t, θ_s and θ are the temperatures of the shield and of the sample, respectively, and K is a constant of the apparatus.

Since the specific heat at constant pressure is given by $C_p = \frac{dH}{d\theta}$, equation (1) may be written in the form

$$C_p \frac{d\theta}{dt} = K(\theta_s - \theta). \qquad (2)$$

We shall further assume that the temperature of the shield is being raised or lowered at a constant rate, i.e., that $\frac{d\theta_s}{dt}$ is a positive or a negative constant.

PROPOSITION I. *If C_p is constant with respect to the temperature, $\theta_s - \theta$ rapidly approaches the constant value $\frac{C_p}{K}\frac{d\theta_s}{dt}$.*

PROOF:

Let $\theta_s - \theta = u$ and $\frac{d\theta_s}{dt} = a$

Then $a - \frac{d\theta}{dt} = \frac{du}{dt}$, or $\frac{d\theta}{dt} = a - \frac{du}{dt}$.

Replacing $\theta_s - \theta$ and $\frac{d\theta}{dt}$ in equation (2) by their values, and rearranging the terms, we obtain

$$\frac{du}{dt} + \frac{K}{C_p}u = a \tag{3}$$

which is a linear differential equation of the first order whose solution is

$$u = e^{-\int \frac{K}{C_p} dt}\left[\int e^{\int \frac{K}{C_p} dt} a dt + C\right]$$

where C is the constant of integration.

But since K, C_p and a are constants, we have

$$\begin{aligned} u &= e^{-\frac{Kt}{C_p}}\left[a\int e^{\frac{Kt}{C_p}} dt + C\right] \\ &= e^{-\frac{Kt}{C_p}}\left[\frac{aC_p}{K}e^{\frac{Kt}{C_p}} + C\right] \\ &= \frac{aC_p}{K} + Ce^{-\frac{Kt}{C_p}}. \end{aligned}$$

Therefore, $\theta_s - \theta = \frac{C_p}{K}\frac{d\theta_s}{dt} + Ce^{-\frac{Kt}{C_p}}$. (4)

In the usual set-up $\frac{K}{C_p}$ is such that $0.693\frac{C_p}{K}$, the half value period of the exponential $e^{-\frac{Kt}{C_p}}$, is about two minutes. Thus, $\theta_s - \theta$ rapidly approaches the constant value $\frac{C_p}{K}\frac{d\theta_s}{dt}$. That is, the rate of temperature rise of the sample approaches a steady state where $\frac{d\theta}{dt} = \frac{d\theta_s}{dt}$.

Figure 1 shows this behavior graphically for a time-temperature curve where $\frac{d\theta_s}{dt}$ is positive, i.e., for a heating curve. The upper broken line is the time-temperature curve of the shield. The full curves are the time-temperature curves of the sample, the lower one being for a case where the value of C in equation (4) is positive, the upper one where it is negative. The lower broken line shows the condition approached by these two curves, a condition in which $\theta_s - \theta$ is constant and equal to $\frac{C_p}{K}\frac{d\theta_s}{dt}$. It will be noted that the numerical value of C is the difference between the θ intercepts of this limiting line and of the curve for the sample; for

$$C = \left[\left(\theta_s - \frac{C_p}{K}\frac{d\theta_s}{dt}\right)_{t=0} - \theta_{t=0}\right]$$

and $\frac{C_p}{K}\frac{d\theta_s}{dt}$ is the vertical distance between the broken lines.

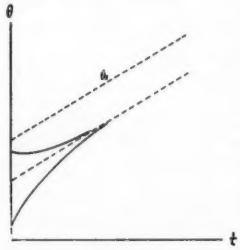

FIGURE 1

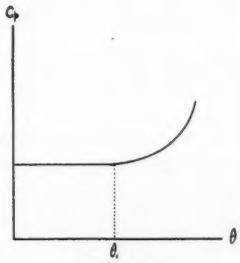

FIGURE 2

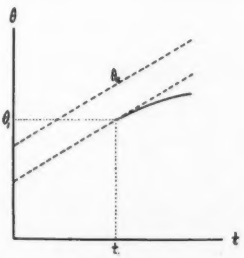

FIGURE 3

PROPOSITION II. *For the case where* $\frac{d\theta_s}{dt}$ *is a positive constant (a heating curve) if* C_p *is constant for a sufficiently long period to attain the steady state mentioned in Proposition I, and if at some later time,* t_1, C_p *starts to increase with the temperature as shown in figure 2, the curve becoming concave upward at* θ_1 *(corresponding to* t_1*), it follows that the temperature curve for the sample is concave downward for all values of t greater than* t_1, *as shown in figure 3.* That is, if the following conditions hold: $\frac{d^2C_p}{d\theta^2} = 0$ when $t < t_1$; $\frac{dC_p}{d\theta} = 0$, $u = \frac{C_p}{K}\frac{d\theta_s}{dt}$ and $\frac{du}{dt} = 0$ when $t \leq t_1$; but $\frac{d^2C_p}{d\theta^2} > 0$ when $t \geq t_1$; then for all values of $t > t_1$, $\frac{d^2u}{dt^2} > 0$.

PROOF:

Differentiating equation (3), which applies here also, we have

$$\frac{d^2u}{dt^2} = \frac{K}{C_p^2}\left(u\frac{dC_p}{dt} - C_p\frac{du}{dt}\right)$$

so that when $t = t_1$, $\frac{d^2u}{dt^2} = 0$ since

$$\frac{dC_p}{dt} = \frac{dC_p}{d\theta}\frac{d\theta}{dt} = 0.$$

Further $$\frac{d^3u}{dt^3} = \frac{K}{C_p^2}\left\{u\frac{d^2C_p}{dt^2} - \frac{d^2u}{dt^2}C_p\left(1 + \frac{2}{K}\frac{dC_p}{dt}\right)\right\}.$$

But $$\frac{dC_p}{dt} = \frac{dC_p}{d\theta}\frac{d\theta}{dt}.$$

Whence, $$\frac{d^2C_p}{dt^2} = \frac{dC_p}{d\theta}\frac{d^2\theta}{dt^2} + \left(\frac{d\theta}{dt}\right)^2\frac{d^2C_p}{d\theta^2}$$

$$= \left(\frac{d\theta}{dt}\right)^2\frac{d^2C_p}{d\theta^2} - \frac{dC_p}{d\theta}\frac{d^2u}{dt^2} \quad \text{since } \frac{d^2\theta_s}{dt^2} = 0.$$

Consequently, $\frac{d^3u}{dt^3} = \frac{K}{C_p^2}\left\{u\left(\frac{d\theta}{dt}\right)^2\frac{d^2C_p}{d\theta^2} - \frac{d^2u}{dt^2}\left(C_p + u\frac{dC_p}{d\theta} + \frac{2C_p}{K}\frac{dC_p}{d\theta}\frac{d\theta}{dt}\right)\right\}.$

Thus, when $t = t_1$, $$\frac{d^3u}{dt^3} = \frac{Ku}{C_p^2}\left(\frac{d\theta}{dt}\right)^2\frac{d^2C_p}{dt^2} > 0.$$

Therefore, when t increases from t_1, $\frac{d^2u}{dt^2}$ becomes positive, also $\frac{d^2u}{dt^2}$ must remain > 0, for if $\frac{d^2u}{dt^2}$ is or becomes less than the positive function,

$$\frac{u\left(\frac{d\theta}{dt}\right)^2\frac{d^2C_p}{d\theta^2}}{C_p + u\frac{dC_p}{d\theta} + \frac{2C_p}{K}\frac{dC_p}{d\theta}\frac{d\theta}{dt}},$$

$\frac{d^3u}{dt^3} > 0$ and $\frac{d^2u}{dt^2}$ is increasing. Thus, the curve being continuous, $\frac{d^2u}{dt^2}$ is greater than zero for all values of t greater than t_1.

It can easily be shown from the foregoing equations that if in Proposition II an insufficient time is allowed for the time-temperature curve of the sample to attain the steady state mentioned in Proposition I before C_p becomes variable: (A) if $C > 0$ the time-temperature curve being already concave downward becomes even more so at $t = t_1$, and (B) if

$C < 0$ the curve being already concave upward, will begin to flatten out more rapidly at $t = t_1$, and will soon become concave downward.

The experimental significance of these propositions will be discussed further in a forthcoming paper.

[1] Skau, *Proc. Amer. Acad. Arts and Sci.*, **67**, 551 (1933).

THE LINKAGE RELATIONS OF YELLOW FAT IN RABBITS

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The fat of ordinary domestic rabbits is white, like the fat of sheep and pigs, but in certain breeds of rabbits an occasional individual is found, when slaughtered, to have yellow fat. This is considered an undesirable characteristic, as it renders the appearance of the carcass unusual and so open to objection on the part of the purchaser. Beef with yellow fat is not unusual and so not objected to by the public.

The color of yellow fat in rabbits is due to an element in their food (carotene, found in yellow carrots, or xanthophyl, occurring in the chlorophyl of green plants) which, being fat soluble, is deposited in the fat of the rabbit. Ordinary white-fat rabbits, according to the biochemists of Cambridge University, England, have in their livers an enzyme which reduces the assimilated carotene to a colorless state. This reducing mechanism is lacking in yellow-fat rabbits, and so the carotene passes unreduced and yellow into the fat storage tissue. The inheritance of the yellow fat character was first studied and described by Pease (1927),[1] who found it occurring sporadically in a race of Flemish Giant rabbits in Cambridge, England. He discovered the character to be a simple recessive in inheritance but genetically linked with albinism—that is, borne in the same chromosome as the color gene, *C*, and its allelomorphs. Some years previously I had shown that a loose linkage exists between the color gene (*C*) and brown hair and skin pigmentation (*b*), with 35.8% crossing-over in a total of 1006 back-cross young.[2] It was to be expected, therefore, that as yellow fat (*y*) is linked with *C*, it should also be linked with *b*. Evidence that this is so will be presented in this paper.

In some preliminary experiments[3] on the inheritance of yellow fat, which I found occurring in three different stocks of our rabbits, I was able to verify Pease's discovery that this character is a good recessive and linked with albinism, and to show that it is also linked with the albino allelomorph chinchilla, with a crossover percentage of 7.8 ± 4.7, an esti-

mate based on a total of 51 back-cross young. It turns out, from the study of larger numbers, that this estimate was too low, as the results of Pease also indicate.

My first task was to obtain a yellow-fat individual suitable for making crosses to test linkage simultaneously between *y* and *C* and between *y* and *b*. For this purpose I chose a non-agouti chocolate chinchilla male having yellow fat (as determined by pedigree and by sample of subcutaneous fat secured from the neck between the shoulder blades). In appearance he was a Havana (chocolate) rex animal, but instead of carrying the factor for full color (*C*), he carried its recessive chinchilla allelomorph (c^{ch}). His genetic formula was accordingly $c^{ch}c^{ch}yybb$, a triple recessive homozygote. He was crossed with typical standard-bred Himalayans which are homozygous for different allelomorphs of all three genes. They are $c^{H}c^{H}YYBB$ in formula and have black pigmented extremities and white fat. The F_1 young were dark sables, only a little lighter than full-colored blacks. They were triple heterozygotes, $c^{ch}c^{H}YyBb$ in formula. The reason for employing Himalayans rather than ordinary albinos (which are completely colorless) in making the cross was to be able readily to ascertain whether the expected segregating albinos carried *B* or *b*. This one can determine, in the case of Himalayans, merely by inspection of the colored extremities, whereas in the case of complete albinos a breeding test would be required.

A triple recessive race had meanwhile been synthesized which could be used in making backcrosses of the F_1 animals. These triple recessives were of the constitution $c^{H}cyybb$ or *ccyybb*, either of which would be equally serviceable, since gametes of the F_1 animals would always carry a color allelomorph, c^{ch} or c^{H}, capable of disclosing the presence of *B* rather than *b* by the colored coat or the colored extremities in the respective classes of back-cross animals.

The backcrosses between the F_1 triple heterozygotes and the triple recessives have thus far produced 477 young, which fall into eight phenotypes as follows:

HIMALAYAN				SABLE			
BLACK		BROWN		BLACK		BROWN	
WHITE FAT	YELLOW FAT	WHITE FAT	YELLOW FAT	WHITE FAT	YELLOW FAT	WHITE FAT	YELLOW FAT
151	2	67	23	33	48	11	142
$c^{H}YB$	$c^{H}yB$	$c^{H}Yb$	$c^{H}yb$	$c^{ch}YB$	$c^{ch}yB$	$c^{ch}Yb$	$c^{ch}yb$

The type of gamete which must have been produced by the F_1 parent to give the combination observed, is indicated for each class. If there were no linkage between these three genes, the eight phenotypes should all be numerically equal, except for effects of random sampling. Their divergence from equality is the evidence for genetic linkage.

The original combinations in the parents of the F_1 animals were $c^{ch}yb$ and $c^{H}YB$. These, then, constitute the *non-crossover classes* in the back-cross generation. They number, respectively, 142 and 151, their sum being 293. The combinations $c^{ch}YB$ and $c^{H}yb$ would result as *single cross-overs between C and Y*. They number 33 and 23, respectively, their sum being 56. From *single crossovers between Y and B* would derive the combinations $c^{ch}yB$ and $c^{H}Yb$, which number, respectively, 48 and 67, their sum being 115. From *double crossovers*, i.e., from crossovers occurring simultaneously between C and Y and between Y and B, would arise the combinations $c^{ch}Yb$ and $c^{H}yB$, which number 11 and 2, respectively, their sum being 13.

To get, therefore, the total number of crossovers occurring between C and Y, we must add 13 to the single crossover total, 56, which brings it up to 69, or 14.4 per cent of the total population, 477. Likewise, to get the total number of crossovers between Y and B we must add 13 to the single crossover total, 115, bringing it up to 128, or 26.8 per cent of the total population, 477.

The apparent number of crossovers between C and B will include all classes in which those two genes have changed their original relations to each other. Those relations were $c^{ch}b$ and $c^{H}B$, respectively. All classes in which these relations persist will be non-crossover classes so far as those genes are concerned. They number, respectively, 151, 2, 11 and 142, their sum being 306. The remaining classes (in which the indicated associations are changed, the four middle ones of the table) number respectively 67, 23, 33 and 48, their sum being 171, which is 35.8 per cent of the total population, 477. This is in exact agreement with the previously determined crossover percentage between C and B, based on a population of 1006 individuals, and is therefore probably highly reliable.

It now becomes obvious what is the order of the three genes. C and B are widest apart, 35.8 being the crossover percentage between them. C and Y are closest together, 14.4 being the crossover percentage. Y lies between C and B, nearer to C than to B, the respective distances being 14.4 and 26.8, respectively. The sum of these two is 41.2, the map distance for the chromosome so far as its genes are known. The difference between this number and the observed crossover percentage between C and B (35.8) is 5.4, which results from the occurrence of double crossovers in 13 instances, these however not affecting the relationship of the two end genes (B and C) to each other, but only their relations to the intervening gene (Y). Hence the apparent amount of crossing-over between B and C (when we consider only their relations to each other) is less than that actually known to have occurred by some 5.4 per cent.

Is the amount of double crossing-over in the two regions (C–Y and Y–B) as great as we should expect if crossing-over in each region occurs quite

irrespective of crossing-over in the other? No, there occurs "interference" in the rabbit chromosome as well as in chromosomes of Drosophila. Chance coincidence of crossing-over in the two regions should occur as the product of their respective frequencies or $14.4 \times 26.8 = 3.8$ per cent. This would call for the occurrence of 18 double crossovers in a population of 477 individuals, whereas only 13 have been recorded.

Summary.—Linkage studies on rabbits involving simultaneously the three genes *C*, *Y* and *B*, show (1) that all three genes are borne in the same chromosome; (2) that the order of the genes is *CYB;* (3) that the crossover percentage between *C* and *Y* is 14.4 ± 1.0 per cent; (4) that the crossover percentage between *Y* and *B* is 26.8 ± 1.4 per cent; and (5) that the crossover percentage between *C* and *B* is (in complete agreement with earlier observations) 35.8 ± 0.8 per cent. Double crossing-over occurs in less than the frequency expected on the theory of probability, hence "interference" is demonstrated for a mammalian chromosome.

[1] Pease, M. S., *Zeit. ind. Abst.-Vererb.*, Bd. **46**, also *Supplementband*, **2**.
[2] Publ. No. 337, Carnegie Inst. Wash., 1926.
[3] *Zeit. ind. Abst.-Vererb.*, **52**, 1929.

RÔLE OF THE AUTOSOMES IN THE DROSOPHILA PSEUDOOBSCURA HYBRIDS

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Lancefield[1] discovered that the species Drosophila pseudoobscura consists of two races, called race *A* and race *B*, respectively. Completely sterile males and partially fertile females appear in the offspring if the races are intercrossed. Males coming from the cross, $B♀ \times A♂$, have rudimentary testes that are smaller in size than the testes of normal males. Testes of the $A♀ \times B♂$ hybrid males are normal in size but incapable of producing functional sperm. Lancefield[1] and Koller[2] have studied the localization of the factors responsible for the sterility of the hybrids; they backcrossed hybrid females to males of the parental races, and observed the testis size and the sterility (vs. fertility) of the male offspring. The results are summarized by Lancefield in the following statement: "a son, who received from his mother an *X*-chromosome derived from the same race as his father, usually had testes of normal or nearly normal size, while a son who received the other *X*-chromosome from his mother usually had small testes." Males possessing an *X*-chromosome of one race and a majority of the autosomes and the *Y*-chromosome of the other race are, thus,

sterile, and usually have small testes. Whether the *Y*-chromosome, or the autosomes, or both, are concerned in the production of sterility cannot be determined from the data of Lancefield and Koller, since any back-cross male necessarily possesses the *Y*-chromosome and a majority of the autosomes of the same race. More critical data may be obtained by studying hybrid males possessing no *Y*-chromosome at all (*XO* males).

Such males were observed in pure race *A* by Schultz.[3] They had testes of normal size and structure, but proved to be sterile. The sterility of these non-hybrid *XO* males has, however, nothing to do with that of the hybrid males. According to Shen,[4] the sterility of the *XO* males in Drosophila melanogaster is due to an abnormal function of the vas efferens rather than to that of the testis itself. Schultz[3] observed also F_1 hybrid *XO* males, and found them to have small testes.

Through non-disjunction of the *X*-chromosomes at the reduction division in hybrid females, males having no *Y*-chromosome may be obtained in the backcrosses of the hybrid females to normal males. Various *B* race strains (Seattle-4 and Seattle-6) were crossed to race *A* strains (La Grande-2 and Texas). The F_1 hybrid females were backcrossed to race *A* males carrying in their *X*-chromosome the dominant gene Pointed (*P*). Regular offspring expected to appear in the next generation are Pointed (*XX*) females and wild-type (*XY*) males. Exceptional offspring (due to non-disjunction of the *X*-chromosomes in the mother) are wild-type (*XXY*) females and Pointed (*XO*) males. These four classes were, indeed, observed (table 1).

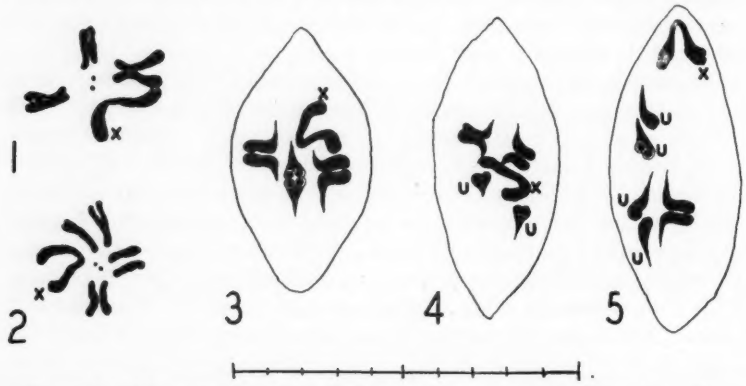
TABLE 1

OFFSPRING OF THE CROSS HYBRID ♀ × POINTED ♂

PARENTAGE OF THE MOTHER	P ♀	WILD ♂	WILD ♀	P ♂
Seattle-6 ♀ × La Grande-2 ♂	2306	1204	3	10
Seattle-4 ♀ × La Grande-2 ♂	1387	781	2	7
Seattle-4 ♀ × Texas ♂	1760	1095	5	8
Texas ♀ × Seattle-4 ♂	89	54	..	3
Total	5542	3134	10	28

Some of the wild-type (*XY*) males shown in table 1 had very small testes, others had testes of normal or intermediate size. This is the normal result observed in the backcrosses of hybrid females to normal males. More important for our purposes is that the Pointed (*XO*) males manifested the same degree of variability of the testis size as their wild-type sibs. Some of these *XO* males were studied cytologically. Numerous spermatogonial division figures were observed; in every figure a large *V*-shaped *X*-chromosome, and no *Y*-chromosome, was found (Figs. 1 and 2), proving that these males were actually *XO*. The histological structure of the testes was, however, as variable as their size. Some of the *XO* males possessing testes of normal size have apparently normal spermatogonia and first sper-

matocytes. At meiosis all the autosomes pair regularly, but the X-chromosome, having no partner, is left as a univalent (Fig. 3). The first meiotic division is normal. The X-chromosome, undivided though showing the equational split, sometimes passes precociously to one of the poles; sometimes on the contrary, it lags on the spindle after the autosomes have disjoined. Two kinds of second spermatocytes are observed: some carry the X-chromosome, and others carry no heterochromosomes. The second meiotic division is normal, and normal spermatids are produced. The mature spermatozoa are, however, distinctly irregular, having short worm-shaped heads instead of the very long and slender ones. Sometimes the



Figures 1 and 2, spermatogonial metaphase plates of the hybrid XO males. Figures 3 to 5, metaphase or early anaphase of the first meiotic division in the hybrid XO males. X, the X-chromosome; U, the univalent autosomes. The scale represents 10 micra.

head is even egg-shaped, the chromatic substance composing it being filled with large vacuoles.

In a number of XO males having testes of normal size the spermatogenesis was found to be disturbed in a manner similar to that observed in XY males coming in the F_1 generation of the cross, $A\,♀ \times B\,♂$ (spermatogenesis in these latter males is described in a preliminary form by Dobzhansky[5] and Dobzhansky and Boche[6]; a more detailed description is in progress). Some or all of the autosomes fail to pair at meiosis (in Fig. 4 two univalents are present, in Fig. 5 one can see four univalents). The first meiotic division is abnormal, the spindle elongates enormously at telophase, bends in a nearly complete ring, two or more daughter nuclei are formed, the cell body fails to divide. The second meiotic division is lacking, giant binucleated spermatids are produced, in which the nuclei lie in the part of the cell directed toward the center of the testis instead of toward the periphery, as is the case in normal spermatids. It is interesting to note that these ab-

normal spermatids are frequently formed in the hybrids from the spermatocytes in which all the chromosomes are paired completely. This fact suggests that the abnormal behavior of the spindle in the hybrids is not directly caused by a failure of chromosome pairing.

XO males having small testes show disturbances in the spermatogenesis similar to those observed in the *XY* males coming from the cross $B\,♀ \times A\,♂$ (Dobzhansky and Boche[6]). The number of spermatogonia and of the first spermatocytes in such testes is much smaller than in normal testes. Chromosomes mostly fail to pair at meiosis. The first meiotic division is usually abortive, since the nucleus as well as the cell body frequently fail to divide (sometimes, however, the nucleus divides into several daughter nuclei of unequal size). The second division is absent, and the abnormal spherical or crescent-shaped spermatids that are formed finally disintegrate.

The *XO* males show, thus, all the degrees of the degeneration of the process of spermatogenesis, ranging from a practically normal course of this process, through the condition found in the $A\,♀ \times B\,♂$ F_1 hybrid males, and to the very profound disturbance observed in the $B\,♀ \times A\,♂$ F_1 hybrid males. The same great variability is shown by the *XY* hybrid males resulting from backcrosses. The genetic structure of all the *XO* males obtained in the progeny of any one cross (table 1) is similar in that they possess the same *X*-chromosome (derived from their father), and the same cytoplasm (derived from their mother and grandmother). They differ from each other only in that some of them possess more autosomal material of race *A* than others. It follows that the variability of the size and structure of the testes of *XO* hybrid males is due to the interactions between the *X*-chromosome of race *A* and the autosomes of race *B*. Whether or not the *Y*-chromosome has any effect on the hybrid testes remains an open question, though on the basis of the data presented above one may surmise that its effect, if any, is small as compared with that of the autosomes and the *X*-chromosomes.

[1] Lancefield, D. E., "A Genetic Study of Crosses of Two Races or Physiological Species of Drosophila obscura," *Zeits. ind. Abst. Verebungsl.*, **52**, 287–317 (1929).

[2] Koller, P. Ch., "The Relation of Fertility Factors to Crossing-Over in the Drosophila obscura Hybrids," *Ibid.*, **60**, 137–151 (1932).

[3] Mentioned in T. H. Morgan, C. B. Bridges and J. Schultz, "The Constitution of the Germinal Material in Relation to Heredity," *Carnegie Institution Year Book*, **29**, 352–359 (1930).

[4] Shen, T. H., "Zytologische Untersuchungen über Sterilität bei Männchen von Drosophila melanogaster und bei F_1 Männchen der Kreuzung zwischen D. simulans Weibchen und D. melanogaster Männchen," *Zeits. Zellforschung mikr. Anat.*, **15**, 547–580 (1932).

[5] Dobzhansky, Th., "On the Sterility of the Interracial Hybrids in Drosophila pseudoobscura," *Proc. Nat. Acad. Sci.*, **19**, 397–403 (1933).

[6] Dobzhansky, Th., and Boche, R. D., "Intersterile Races of Drosophila pseudoobscura," *Biol. Zentralblatt*, **53**, 314–330 (1933).

AN EOCENE PRIMATE FROM CALIFORNIA

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Introduction.—Recognition of a primate in the mammalian assemblage now being recorded from one of the faunal stages of the Sespe, north of the Simi Valley, California, comes with startling abruptness, yet is not wholly unexpected. No fewer than five fragments of jaws, probably representing as many individuals, have been found recently. The sudden appearance of this material in the collections of fossil mammalian remains obtained at Locality 150 Calif. Inst. Tech. Vert. Pale. commands attention when it is stated that excavations at this site have been in progress more or less continuously during the past two years.

Relationship of the new type appears to be particularly close with that division of the tarsiid lemurs (Anaptomorphidae) which includes the Eocene genera *Omomys*, *Hemiacodon* and *Euryacodon*. Lending interest to the discovery is the fact that this form represents the latest primate to be recorded from the North American Tertiary. Moreover, the occurrence marks another outpost for a group of mammals whose presence on this continent during the Eocene has been identified heretofore with intermontane basins of the Cordilleran region.

Chumashius balchi, n. gen. and n. sp.

Type Specimen.—Fragmentary left ramus of mandible with $P\bar{3} - M\bar{3}$ and alveoli for $P\bar{2}$ and C, No. 1391 C. I. T. Vert. Pale., plate 1, figures 1, 1*a*.

Paratypes.—An incomplete left mandibular ramus, No. 1392, plate 1, figures 3, 3*a* and 3*b*, and a fragment of palate with $M\underline{2}$ and $M\underline{3}$, No. 1394, plate 1, figure 4.

Referred Specimens.—Two incomplete mandibular rami with portions of the premolar-molar series, No. 1393, plate 1, figures 2 and 2*a*, and No. 1390.

Locality.—Sespe Upper Eocene, north of Simi Valley, Ventura County, California; Locality 150 C. I. T. Vert. Pale.

Generic and Specific Characters.—Dentition $\bar{2}?, \bar{1}, \bar{3}, \bar{3}$. $P\bar{4}$ without distinct metaconid. Paraconid median and distinct in $M\bar{1}$, more reduced in $M\bar{2}$, vestigial in $M\bar{3}$. $M\bar{3}$ reduced in size and with inner and posterior cusps of heel subdued or absent. Enamel smooth. $M\underline{2}$ and $M\underline{3}$ with well developed cingula, except on external sides. Intermediate cuspules small and faintly indicated. $M\underline{3}$ more reduced in transverse diameter than in *Euryacodon*. No indication of antero-internal cuspules on $M\underline{2}$ and $M\underline{3}$ as in *Euryacodon*. Referred specimens larger than type and paratype and

approximate in size the species *Omomys carteri*. The species is named for Mr. Allan C. Balch.

Description.—Although the maxillary fragment bearing the two upper molars cannot be directly associated with any one of the jaws, the presence of probably four individuals of this primate at locality 150, as based upon the lower jaw material, makes it reasonable to assume that all of the specimens pertain to the same type of creature. Moreover, some of the characters exhibited by the upper teeth are to be correlated with those displayed by the lower posterior molars. The four lower jaws from locality 150 show some variation in size (compare for example Figs. 1, 1*a* and 2, 2*a*). Judging from the state of wear of the teeth in these specimens, the difference in size cannot be accounted for on the basis of age. With exception of the larger size of $M\bar{3}$, relative to the anterior molars, in No. 1393, no important structural features appear to distinguish this specimen from the type. On the other hand, the differences in size may reflect a difference in sex. In this connection it may be mentioned that the canine alveolus in the referred ramus, No. 1390, suggests the presence of a tooth of comparatively large size.

In *Chumashius* the mandibular rami were not fused at the symphysis. The horizontal ramus, as seen in No. 1391, is straight and sturdy. Two mental foramina are present, the posterior situated beneath $P\bar{3}$ and the anterior below and slightly in front of the alveolus for $P\bar{2}$. The anterior base of the ascending portion of the ramus is broad but thins rapidly in its upward course. The anterior portion of the masseteric area is best shown in the paratype, No. 1392, where it is seen to be quite shallow. The posterior dental foramen can be discerned on the inner side of the ascending portion of the ramus (Fig. 3*b*, *p.d.f.*). The position of this opening is closer to the lower border than to the upper anterior border. Fortunately the angle is preserved in this specimen. A low, subdued crest is present on the inner surface of the angle. The process extends downward and backward and is hook-like in shape. The latter character in *Chumashius balchi* is totally unlike that in *Tarsius* and somewhat resembles that in modern lemurs other than *Tarsius* as well as in *Notharctus*.

The present comparisons of the dentition in the Sespe specimens may be found somewhat limited in scope, in view of the fact that the available comparative materials of primates include only a cranium and mandible of *Tarsius spectrum* borrowed from the U. S. National Museum and the type specimen of *Omomys carteri* Leidy kindly loaned to me by the Academy of Natural Sciences of Philadelphia. However, published descriptions and illustrations of Eocene primates, particularly those by Wortman,[1] Matthew[2] and by Gregory[3] permit an extension of the observations made on the forms from the Simi locality.

Although the lower dentition preserved in the materials from the Simi

PLATE 1. (Description on opposite page.)

gives a representation of $P\bar{3} - M\bar{3}$, three of the four jaws furnish evidence of value in determining the lower dental formula. In the type specimen, No. 1391, as well as in No. 1393, two distinct alveoli are to be seen anterior to $P\bar{3}$. These are presumably for the canine and premolar 2. In both jaws the alveolus for the canine is larger than that for $P\bar{2}$. Similar characters are to be noted in No. 1390. Fortunately, in this specimen more of the anterior end of the jaw is preserved although the tip is not present. Portions of the alveolar walls of at least two incisors can be discerned. Thus, in number of lower teeth *Chumashius* differs from the Eocene genera *Anaptomorphus* and *Tetonius* and resembles *Omomys* and *Hemiacodon*.

$P\bar{3}$ resembles the comparable tooth in *Omomys* in shape of base and height of crown. A rudimentary posterior ledge is present and a metaconid is absent.

$P\bar{4}$ likewise resembles the comparable tooth in *Omomys* in shape of base and height of crown. It differs, however, from the fourth premolar in *O. carteri* in absence of a distinct metaconid. In the three specimens from the Simi where the crown of this tooth is preserved either entirely or in part, the development of the metaconid is seen to be subject to some variation. In both the type and No. 1393 a slight bulging of the postero-internal border of the protoconid is all that denotes the presence of a metaconid.

Greater definition of this cusp is seen in No. 1390, although in this specimen also the metaconid is not so distinctly formed as in the type of *O. carteri*. In tendency to reduce the metaconid the Sespe species appears to be more advanced than *O. carteri*. A narrow ledge extending the full width transversely forms the posterior base of the tooth. Toward the outer side the surface of this base as well as that on the lower flank of the protoconid immediately adjacent are worn, due to occlusion with an upper tooth. Both $P\bar{3}$ and $P\bar{4}$ are shorter and wider than the comparable teeth in *O. carteri*.

$M\bar{1}$ and $M\bar{2}$ are likewise shorter and wider than in the latter species. In at least one specimen, No. 1390, the basins of the talonids of the anterior molars appear to be shallower than those of the Bridger species. The anterior molars show the characteristic widening of the talonid region seen in *Omomys*. In $M\bar{1}$ (see figure of paratype, plate 1, Fig. 3*a*) the paraconid is well developed, and the trigonid region resembles that in the genotype of *Omomys*.

DESCRIPTION OF PLATE 1

Chumashius balchi, n. gen. and n. sp.

Figures 1 and 1*a*, type specimen, No. 1391, left ramus with $P\bar{3}$–$M\bar{3}$ and alveoli for C and $P\bar{2}$, lateral and superior views; figures 2 and 2*a*, referred specimen, No. 1393, lateral and superior views; figures 3, 3*a* and 3*b*, paratype, No. 1392, lateral, superior and inner views, respectively: figure 4, paratype, No. 1394, maxillary fragment with $M\bar{2}$ and $M\bar{3}$, occlusal view. All figures × 4.

California Institute of Technology Collection. Sespe Upper Eocene, California.

In the second lower molar the paraconid is not so well defined as in $M\bar{1}$ and is also more reduced. The paraconid is even more reduced or is vestigial in $M\bar{3}$. Thus in the lower molars the trigonid region becomes progressively smaller and the greater fore and aft narrowing of this area, as one proceeds from the first to the last tooth, is particularly striking (Figs. 1*a*, 2*a* and 3*a*). A similar progressive reduction of the paraconid in the molars occurs in *Hemiacodon*, according to Matthew. In the characters noted *Chumashius balchi* differs from *Omomys carteri* and is more advanced than this type. A closer approach to our form appears to be made by specimens described by Wortman and referred to the species *Euryacodon lepidus*.

$M\bar{3}$ in the Simi form is distinctly narrower transversely than either $M\bar{2}$ or $M\bar{1}$ and there is likewise less widening of the talonid region. No inner cusp is present on the talonid and the inner rim swings backward to form the posterior rim of the posterior lobe. The posterior rim is worn in No. 1393 and in No. 1391 and there is no clear indication of a hypoconulid. In No. 1392, however, the faintly scalloped inner rim merges posteriorly with a worn surface marking the presence of a hypoconulid. Wortman has pointed out that in the third lower molar of the jaw referred to *Euryacodon lepidus* the crown is narrower than in *Omomys* and differs also from that in the latter in absence of distinct pointed cusps in the talonid region.

The discrepancy in size between $M\bar{3}$ and the anterior molars is not so evident in the referred specimen, No. 1393, as in the type and paratype of *C. balchi*. Unfortunately, the last molar is lacking in the second referred specimen, No. 1390.

In the second upper molar of the paratype, No. 1394, a well defined cingulum is present on the anterior and posterior sides of the tooth and also where it is preserved along the inner side. A portion of the cingulum on the inner side has been broken away, but sufficient of it remains to indicate that an antero-internal cuspule, such as characterizes *Euryacodon*, was not present. The intermediate cuspules are low and not very distinct.

$M\bar{3}$ is considerably more reduced in transverse diameter than the comparable tooth in *Omomys* or in *Euryacodon*. The tooth is broader internally than that of *E. lepidus*. The anterior intermediate cuspule is vestigial. The enamel surfaces of both upper teeth are smooth.

Relationships.—In two primitive characters, namely, the sutural separation of the mandibular rami at the symphysis and the hook-like angular process, *Chumashius* differs from the modern tarsier. These characters have been noted by Gregory[4] as also distinguishing the earlier species of *Notharctus*. In contrast to the primitive features and small size of the Simi form, *Chumashius* is more advanced than *Tarsius* in tendency to reduce the metaconid in $P\bar{4}$, reduction in size of paraconid in the posterior lower molars with vestigial character of this cusp in $M\bar{3}$, and in reduction in size of the

last upper and lower molars. *Chumashius balchi* cannot, therefore, be regarded as occurring in the direct line of development leading upward to the Recent tarsier.

In lower dental formula *Chumashius* exhibits a closer relationship to *Omomys* and *Hemiacodon* than to *Anaptomorphus* and *Tetonius*. The character of the lower posterior premolars in *Uintanius* suffice to remove this Bridger genus from any close relationship with the Simi form. In addition to the presence of a comparable number of lower teeth the structural details of the dentition, in so far as these are known, point also to a kinship between *Chumashius* and that division of the Anaptomorphidae including *Omomys* and *Hemiacodon*. The characters of the Simi genus suggest a development from those of an antecedent form like *Omomys* or *Hemiacodon*. Possibly *Euryacodon* is also situated close to the stem form from which *Chumashius* has evolved. No previously described tarsiid from the North American Eocene has been found in association with a fauna as advanced as that occurring with *Chumashius balchi* at Locality 150 in the Sespe deposits of the Simi Valley region, California.[5]

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[2] Matthew, W. D., *Bull. Amer. Mus. Nat. Hist.*, **34**, 445–465 (1915).

[3] Gregory, W. K., *Mem. Amer. Mus.*, n. s., **3**, pt. 2 (1920).

[4] Gregory, W. K., *Bull. Geol. Soc. Amer.*, **26**, 419–446 (1915).

[5] Stock, C., *Proc. Nat. Acad. Sci.*, **18**, 518–523 (1932); **19**, 434–440, 481–486, 762–767 (1933).

NOTE ON POLYNOMIAL INTERPOLATION TO ANALYTIC FUNCTIONS

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Let C be a Jordan curve of the z-plane, and let the function $w = \phi(z)$ map the exterior of C onto the exterior of the unit circle $|w| = 1$ so that the points at infinity correspond to each other. Let C_ρ denote generically the curve $|\phi(z)| = \rho > 1$ exterior to C.

If the function $f(z)$ is single-valued and analytic interior to C_ρ but has a singularity on C_ρ, then any sequence of polynomials $p_n(z)$ of respective degrees n is said to converge to $f(z)$ on C with the *greatest geometric degree of convergence* provided for every $R < \rho$ there exists M dependent on R but not on n or z such that we have

$$|f(z) - p_n(z)| \leqq \frac{M}{R^n}, \quad z \text{ on } C. \tag{1}$$

Under the hypothesis made on $f(z)$, it is known that there exists no sequence $p_n(z)$ such that (1) is valid for $R > \rho$.

The concept of greatest geometric degree of convergence is of central importance in the study of polynomial approximation to analytic functions. Many well-known sequences of polynomials, such as those of best approximation in various senses, converge on the given point set with the greatest geometric degree of convergence.[1] Under the hypothesis that the sequence $p_n(z)$ converges to $f(z)$ on C with the greatest geometric degree of convergence, it is true that the sequence $p_n(z)$ converges to $f(z)$ at every point interior to C_ρ, uniformly on any closed point set interior to C_ρ, and converges uniformly in no region containing in its interior a point of C_ρ.

The object of the present note is to prove the following:

Let the points $z_1^{(n)}, z_2^{(n)}, \ldots, z_{n+1}^{(n)}$, $n = 0, 1, 2, \ldots$, lie on or within C or more generally have no limit point exterior to C. Let $p_n(z)$ denote the polynomial of degree n found by interpolation to $f(z)$ in the points $z_\kappa^{(n)}$. Then a necessary and sufficient condition that for every $f(z)$ analytic on and within C the sequence $p_n(z)$ should converge to $f(z)$ on C with the greatest geometric degree of convergence is

$$\lim_{n\to\infty} \left|(z - z_1^{(n)})(z - z_2^{(n)}) \ldots (z - z_{n+1}^{(n)})\right|^{1/(n+1)} = r\left|\varphi(z)\right|, \qquad (2)$$

for z exterior to C, where r is the Robin's constant (capacity or transfinite diameter) of C.

If the points $z_1^{(n)}, z_2^{(n)}, \ldots, z_{n+1}^{(n)}$ are not all distinct, the condition of interpolation naturally requires agreement of suitable derivatives of $p_n(z)$ and $f(z)$ at the multiple points.

We introduce the notation

$$\omega_n(z) = (z - z_1^{(n)})(z - z_2^{(n)}) \ldots (z - z_{n+1}^{(n)}).$$

A suitably chosen branch of the function $[\omega_n(z)]^{1/(n+1)}$ is analytic except at infinity exterior to any C_ρ which contains no point $z_i^{(n)}$, and at infinity has the derivative unity with respect to z. For sufficiently large n, each factor $z - z_i^{(n)}$ is uniformly limited in any closed finite region of the plane, and hence the functions $[\omega_n(z)]^{1/(n+1)}$ are similarly limited and form a normal family in the region exterior to C but not containing the point at infinity.

We are here using a slight extension of the usual notion of normal family, for the functions $[\omega_n(z)]^{1/(n+1)}$ are not necessarily analytic at each point exterior to C. Nevertheless, in any closed limited region exterior to C, at most a finite number of those functions fail to be analytic, so the entire set of functions has the characteristic properties of a normal family.

We proceed to prove the sufficiency of condition (2). The validity of condition (2) implies the uniformity of approach to the limit on an arbitrary closed limited point set exterior to C, since the functions $[\omega_n(z)]^{1/(n+1)}$

form a normal family. Let $f(z)$ be analytic interior to C_ρ but have a singularity on C_ρ and let $R < \rho$ be arbitrary. Choose R_1, $R < R_1 < \rho$, and R_2, $1 < R_2 < R_1/R$.

For sufficiently large n, the points $z_k^{(n)}$ lie interior to C_{R_1}, and we have the well-known formula

$$f(z) - p_n(z) = \frac{1}{2\pi i}\int_{C_{R_1}} \frac{\omega_n(z)f(t)\,dt}{\omega_n(t)(t-z)}, \quad z \text{ interior to } C_{R_1}. \tag{3}$$

Thus we have by the use of an obvious inequality,

$$\overline{\lim_{n\to\infty}}\,[\max\,|f(z) - p_n(z)|,\ z \text{ on } C_{R_2}]^{1/(n+1)} \leqq \frac{R_2}{R_1} < \frac{1}{R},$$

for we have uniformly $\lim\,[\omega_n(z)]^{1/(n+1)} = rR_2$, z on C_{R_2}; $\lim\,[\omega_n(t)]^{1/(n+1)} = rR_1$, t on C_{R_1}. This inequality holds for z on C_{R_2} and hence in particular for z on C; we have shown that the sequence $p_n(z)$ approaches $f(z)$ on C with the greatest geometric degree of convergence.

In the proof of the necessity of condition (2) it happens that we need merely study the expansion of the function $f(z) = 1/(t - z)$, where t is suitably chosen exterior to C. For this particular function we have

$$f(z) - p_n(z) = \frac{\omega_n(z)}{\omega_n(t)\,(t-z)};$$

it may be verified directly that $p_n(z)$ as thus defined is a polynomial in z of degree n, and it is seen by inspection that $f(z)$ and $p_n(z)$ are equal in the points $z = z_k^{(n)}$.

By hypothesis the sequence $p_n(z)$ converges to $f(z)$ on C with the greatest geometric degree of convergence. Then we have

$$\overline{\lim}\,\sqrt[n+1]{\frac{M_n}{|\omega_n(t)|}} \leqq \frac{1}{R},\ M_n = \max\,[|\omega_n(z)|,\ z \text{ on } C],\ |\phi(t)| = R.$$

But we have also[2]

$$|\omega_n(t)| \leqq M_n R^{n+1},$$

whence

$$\sqrt[n+1]{\frac{M_n}{|\omega_n(t)|}} \geqq \frac{1}{R}, \qquad \overline{\lim}\,\sqrt[n+1]{\frac{M_n}{|\omega_n(t)|}} \geqq \frac{1}{R}.$$

It follows finally that we have

$$\lim\,\sqrt[n+1]{\frac{M_n}{|\omega_n(t)|}} = \frac{1}{R}.$$

If now t_1 is any other point on the curve C_R, we likewise have

$$\lim \sqrt[n+1]{\frac{M_n}{|\omega_n(t_1)|}} = \frac{1}{R'} \qquad \lim \sqrt[n+1]{\left|\frac{\omega_n(t)}{\omega_n(t_1)}\right|} = 1. \tag{4}$$

Each factor $|z - z_k^{(n)}|$ is uniformly bounded from zero, for n sufficiently large, in any closed region exterior to C. Hence no limit function of the normal family $[\omega_n(z)]^{1/(n+1)}$ can vanish exterior to C. Any limit function of the family is of constant modulus on every C_R, by (4).

Let us choose $\phi(z)$ so that the function $r\phi(z)$ has a Laurent development at infinity of the form $z + a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \ldots$; this is possible by the definition of r. Then the functions

$$\frac{[\omega_n(z)]^{1/(n+1)}}{r\,\phi(z)} \tag{5}$$

are all analytic at infinity (when suitably defined) and take on the value unity at infinity. The functions (5) also form a normal family in the region exterior to C, including the point at infinity, and any limit function of the family is of constant modulus on every C_R, is analytic and different from zero exterior to C, and has the value unity at infinity. Hence, every such limit function is itself unity and (2) is established.

Condition (2), valid uniformly on any closed limited point set exterior to C, implies that the points $z_k^{(n)}$ have no limit point exterior to C.

Our theorem can be extended to point sets C much more general than Jordan regions; it is sufficient if the complement K of C is connected, and regular in the sense that Green's function for K with pole at infinity exists.

If condition (2) is satisfied uniformly on any closed limited point set exterior to C, it will be noted that the lemniscates

$$|\omega_n(z)| = r^{n+1}R^{n+1}$$

approximate uniformly the curve C_R. The reciprocal is also true. Approximation not of curves C_R but of C itself by lemniscates was considered by Faber[3] in connection with the condition

$$\lim_{n\to\infty} p_n(z) = f(z), \text{ uniformly on } C,$$

for an arbitrary $f(z)$ analytic on and within C, but Faber's conclusions require essential modification.

It frequently arises in practice that the polynomials $\omega_n(z)$ are not given directly, but rather polynomials $\psi_n(z)$ which are constant multiples of the $\omega_n(z)$; the coefficient of z^{n+1} is not necessarily unity. Formula (3) is clearly valid if the $\omega_n(z)$ and $\omega_n(t)$ are replaced by $\psi_n(z)$ and $\psi_n(t)$. The condition

$$\lim_{n\to\infty} |\psi_n(z)|^{1/(n+1)} = \sigma |\phi(z)|, \quad z \text{ exterior to } C, \tag{6}$$

uniformly on any closed limited point set exterior to C, where σ is a non-vanishing constant, is also sufficient that (1) should be valid for every (z) analytic on and within C, as follows by the reasoning already used.

It is worth noticing that the validity of (6) ($\sigma \neq 0$) uniformly on any closed limited point set exterior to C implies the corresponding condition (6) where $\psi_n(z)$ is replaced by its derivative $\psi_n'(z)$. This new condition implies that the roots of the $\psi_n'(z)$ have no limit point exterior to C, so the property expressed in our theorem, if satisfied for interpolation in the roots of $\psi_n(z)$, is also satisfied for interpolation in the roots of $\psi_n'(z)$, $\psi_n''(z)$, etc.

There are a large number of illustrations in the literature[3] of the use of the condition of the theorem (taken together with the auxiliary remarks) to prove convergence to $f(z)$ of the sequence $p_n(z)$, ordinarily with less precise results on convergence than we have established here: interpolation in the roots of unity and in certain real points studied by Runge, generalization of interpolation in the roots of unity studied by Fejér and Kalmár, interpolation in the roots of Faber's polynomials and the roots of Tchebycheff polynomials studied by Faber, interpolation in the roots of Szegö's polynomials studied by him, and interpolation in points introduced by Fekete. For the case that C is a line segment or the unit circle, the points $z_k^{(n)}$ may also be chosen as the roots of various orthogonal polynomials.[4]

Our main theorem is clearly related to the following theorem, transmitted to the writer some time ago by Mr. Fekete; the proof is still unpublished:

Let C be an arbitrary Jordan curve, and let the points $z_1^{(n)}, z_2^{(n)}, \ldots, z_{n+1}^{(n)}$ lie on or within C. Then a necessary and sufficient condition that the sequence $p_n(z)$ should converge to $f(z)$ interior to C (not necessarily uniformly) for every $f(z)$ analytic on and within C, is

$$\lim_{n\to\infty} \sqrt[n+1]{M_n} = r, \quad M_n = \max[|\omega_n(z)|, z \text{ on } C].$$

If this condition is satisfied, we have $\lim_{n\to\infty} p_n(z) = f(z)$ uniformly on any closed set interior to C, for every such $f(z)$.

[1] See the writer's forthcoming *Approximation of Polynomials in the Complex Domain*, Mémorial des sciences mathématiques.

[2] Loc. cit. §9.

[3] For detailed references, the reader may refer to Walsh, loc. cit., Ch. V.

[4] Compare the asymptotic formulas given by Szegö, *Math. Ann.*, **82**, 188–212 (1921).

THE INFLUENCE OF INTENSITY ON THE VISUAL FUNCTIONS OF DROSOPHILA

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1. *Purpose and Plan.*—It has been known for many years that the prevailing light intensity determines the magnitude of many visual functions. However, the attempt to formulate these relationships theoretically has made it plain that quantitative comparisons among the data for human vision are uncertain due to important differences in the conditions of their measurement. Because of this, we determined to measure these functions under constant conditions in an animal whose vision is so different from our own as to furnish an independent description of its physiological structure. Such work, already begun with the bee,[1] has been carried further by Wolf.[2] But the seasonal existence of the bee led us to use the common fruit-fly, Drosophila melanogaster, which, like the bee, is genetically uniform, and in addition is available the year round.

The method of measurement, first worked out for the bee, depends on the reflex response given by an animal to a movement of its visual surroundings. Presented with a visual field composed of stripes, the animal can respond to a movement of this pattern only when it is able to resolve the essential elements of the pattern. The composition of the pattern may then be varied to yield the desired visual function. The stripes may be varied in width and thus measure visual acuity; the alternating stripes may be varied in relative intensity and measure intensity discrimination; in spectral composition and measure wave-length sensibility and color vision. The present report concerns measurements of visual acuity and of intensity discrimination.

2. *Procedure.*—As shown in figure 1, light from a 500-watt concentrated filament Mazda lamp falls successively on three opal glass plates. The intensity of the light on opal plate 3 is controlled in a series of steps by placing the lamp at selected distances from opal 1, and by removing or keeping opal 1; for the intervening steps the intensity is controlled smoothly by the iris diaphragm in front of opal 2. Immediately in front of opal 3 is the vertically striped pattern, mounted to slide easily in the plane perpendicular to the drawing. For intensity discrimination the pattern is a photographic plate composed of *translucent* bars and equally wide, clear interspaces. Several similar plates are used which differ only in the density of the translucent bars, so that each plate represents a pattern whose alternating elements have a fixed ratio of light transmission. For visual acuity the pattern is a photographic plate composed of *opaque*

bars and equally wide clear interspaces. Several such plates are used differing only in the width of the stripes. The measurements are made by setting for each plate an intensity at which motion of the plate produces no response. The intensity is then raised by small steps until the characteristic threshold reaction of the animal is elicited.

The fly is in a long, rectangular, glass cell in front of the moving pattern, and presents one eye full toward the pattern. With the light on, and the stripes at rest, the fly creeps back and forth from one end of the cell to the other. When the stripes are moved in the direction in which the fly is creeping, the fly stops, moves backward a few millimeters, turns around and creeps in the opposite direction. To obtain a sharp response it is not necessary for the striped plate to move more than just perceptibly faster

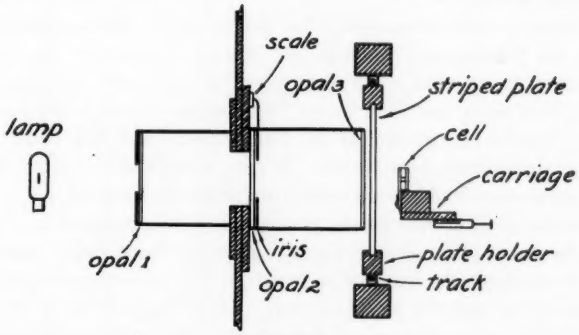

FIGURE 1

Apparatus for measuring intensity discrimination and visual acuity of Drosophila.

than the fly. A much more rapid motion may cause fusion of the stripes, a source of error not recognized by Graham and Hunter[3] in their use of this method for human visual acuity. This behavior of Drosophila is an inherited reflex because a fly raised in the dark responds to its very first light exposure precisely as an older fly raised in the light.

3. *Intensity Discrimination.*—We made 113 measurements of the intensity discrimination of 24 flies. Each determination with each fly is shown in figure 2. The measurements have been averaged in the obvious groups into which they fall, and the curve passes through these averages. The data show that at the lowest intensities, for the fly to respond to the pattern, the ratio of the intensities of alternating stripes must be about 100. As the prevailing intensity increases, this ratio decreases rapidly to about 2.5, a value which is maintained up to the highest intensities. This is shown by the fact that the plot in figure 2 approaches a straight line whose slope is 1. Not shown in figure 2 is the fact that the line continues this

way up to 1000 millilamberts, the highest intensities obtainable in the present apparatus.

It is usual to call I the lower of the two intensities and $I + \Delta I$ the higher, the difference between them being ΔI. As a measure of intensity discrimination, one may then plot either the ratio $\Delta I/I$ against log I, as has been done since Fechner, or preferably the ratio $\Delta I/(I + \Delta I)$ against log $(I + \Delta I)$. Neither procedure has any apparent theoretical justification, and in the present case they tell no more than the data themselves as plotted in figure 2. From the figure it is apparent that below log $(I + \Delta I) = -2.1$ the intensity discrimination of Drosophila is practically non-existent, but that above this it improves rapidly up to about log $(I + \Delta I) = -1.7$, after which it slowly approaches a constancy which is maintained at the highest intensities.

The intensity discrimination of the bee follows a similar course. However for the human eye, Koenig and Brodhun,[4] Lowry[5] and Houstoun and Shearer[6] found that as the intensity rises intensity discrimination at first becomes better and then worse. The same is apparently true for the clam.[7] Aubert's measurements[8] for the human eye do not show this final fall, and unpublished data by Mr. Jacinto Steinhardt of our Laboratory confirm its absence at high intensities under proper conditions.

4. *Visual Acuity.*—We made 220 measurements of visual acuity with 32 flies. The individual points in figure 3 give the visual angle subtended by the stripe to which the bee just responds at the particular intensity. The curve passes through the averaged data. Adopting the usual definition of visual acuity as the reciprocal of the just resolvable visual angle in minutes, we have plotted the averaged data in this way in figure 4. The visual acuity of Drosophila is thus shown to increase with the logarithm of the illumination in a sigmoid manner, already familiar from the data on the human eye and on the bee eye.

The visual acuity of Drosophila does not decrease continuously with the decrease in intensity, but instead stops at an intensity corresponding to a brightness of 0.008 millilambert. No matter how large the stripes are, the animals do not respond to them until this intensity is reached. This is related to the fact, obvious from the intensity discrimination data in figure 2, that at this point for another intensity to be recognized as perceptibly lower it must be practically extinguished.

The maximum visual acuity achieved by Drosophila is 0.0018, a value about $^1/_{1000}$ that of the human eye and $^1/_{10}$ that of the bee's eye. The maximum visual acuity of the human eye and the bee's eye is associated with the size of the structural units of the receiving elements. In man the maximum value as ordinarily obtained approximates the distance between foveal cones. In the bee the minimum perceptible visual angle is the same as the smallest angles (also 0.9–1.0°) subtended by the ommatidia in the

central portion of the eye. In Drosophila the average maximum visual acuity corresponds to a visual angle of 9.28°. Measurements of sections of the eye show that the minimum ommatidial angle is very nearly 4.2°. The maximum visual acuity therefore corresponds to an angle which includes two ommatidia instead of one.

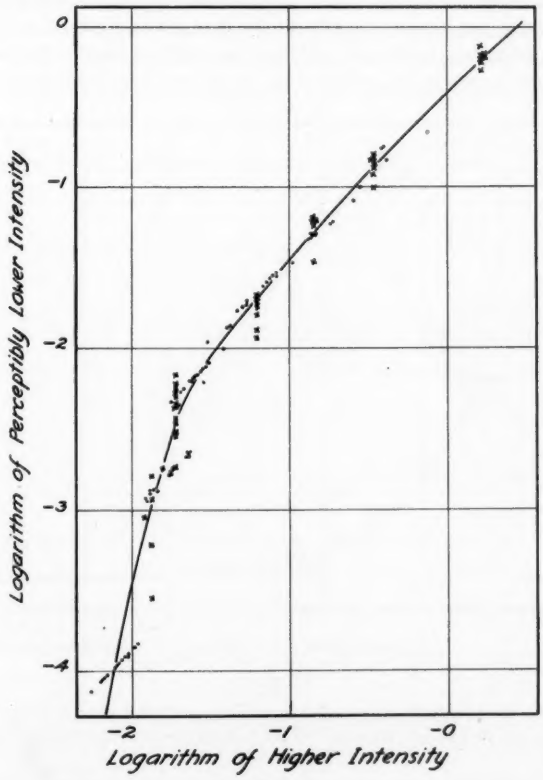

FIGURE 2

Intensity discrimination of Drosophila. Each measurement with 24 flies is recorded. The dots are data secured with the apparatus here described; the crosses with a totally different set-up which it is unnecessary to detail here. The data from the two sources yield the same results.

This might indicate that the ommatidia of Drosophila do not function singly, but we are inclined to attribute the difference to the small total number of ommatidia in the eye. To distinguish a pattern, a certain minimal number of elements must be stimulated. This number is apparently

a small fraction of the total population of retinal elements in the eye of man or of the bee. In the fly where the total number of elements is small, it represents a considerable proportion of the retinal population, and the group of units called into play to register a single stripe thus transcends the boundaries of a single line of elements.

The two visual functions studied in Drosophila occur within about the same intensity range, as is apparent by comparing figure 2 with figure 4. The same holds for the bee as is shown by a comparison of the recent work of Wolf[2] on intensity discrimination with the previous work on visual

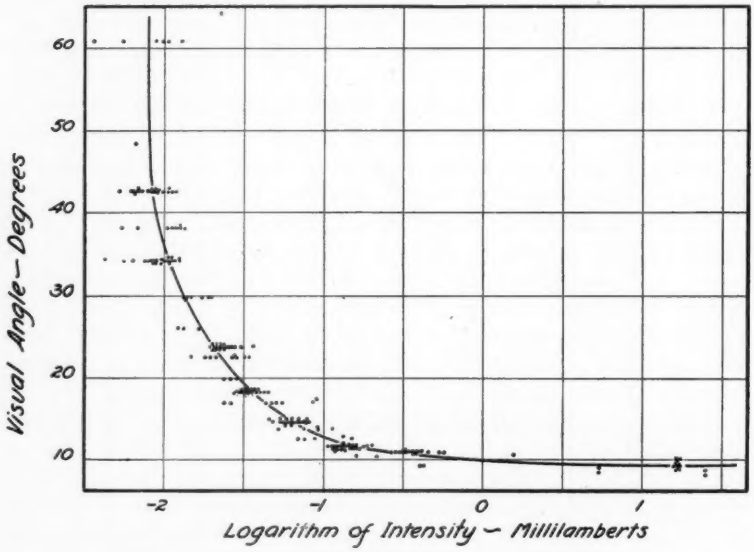

FIGURE 3

The relation between intensity and the angular distance occupied by the stripes to which the fly can just respond. Each measurement with 32 flies is recorded.

acuity.[1] A similar comparison of data for the human eye is not possible because of the different conditions under which they were secured.

The minimum value of $\Delta I/I$ for Drosophila is 1.5; for the bee[2] it is 0.25; and for man the minimum recorded[9] is 0.006. Adopting the entirely arbitrary notion that the reciprocal of $\Delta I/I$ measures intensity discrimination, and putting Drosophila at 1, the ratios Drosophila/bee/man are 1/60/249 for maximum intensity discrimination, and 1/9.4/1110 for maximum visual acuity. A rough parallelism is apparent in these two functions.

5. *Criticisms.*—Drosophila is the fourth organism (the others are man,

the bee and the fiddler crab[10]) whose visual acuity has been measured in relation to intensity, and in all four visual acuity varies with log I in a characteristically sigmoid manner. The only quantitative interpretation at present available[11] for this recognizes that visual acuity is an expression of the resolving power of the retinal surface. The resolving power of a surface composed of independently functioning elements depends on the number of elements per unit area, or more specifically on the distance between the centers of the sensitive elements. To account for the required

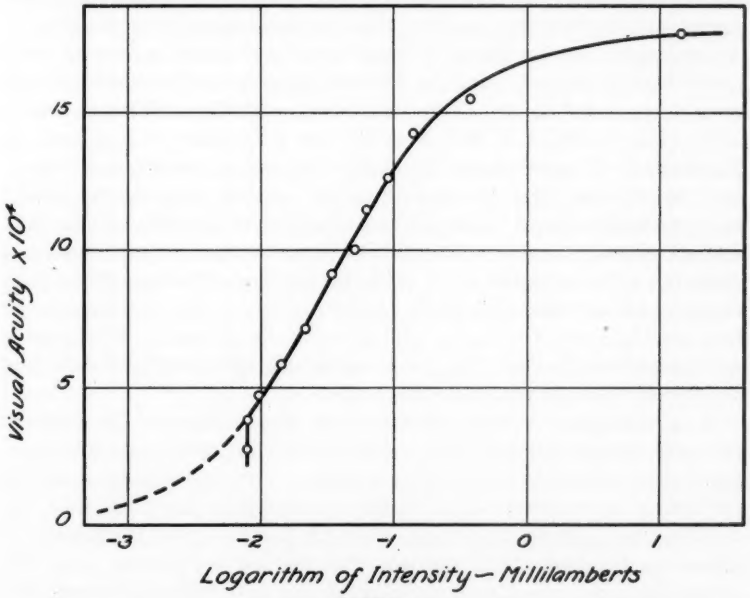

FIGURE 4

The averaged data of figure 3 plotted as visual acuity against the logarithm of the intensity. The function starts abruptly at −2.1, below which the flies do not respond to stripes no matter how large they are.

variation in number of elements at different intensities, it is assumed that the thresholds of the retinal elements vary as does any other population characteristic, and therefore that the visual acuity-log I relation is described by an integral distribution curve.

Freeman[12] has argued that since visual acuity may be varied by factors other than intensity, its variation with intensity cannot be determined by the number of elements functional. A similar argument has been more elaborately presented by Wilcox and Purdy.[13] However, it is hard for us to see the force of these criticisms, since the idea of how visual acuity varies

with intensity is quite independent of the particular mechanism which determines the magnitude of visual acuity at a given intensity. No matter what that mechanism may be or how involved it may be, it must rest on the fact that the ultimate resolving surface is composed of units which are independent functionally. The variation of visual acuity with intensity then follows in terms of the ordinary probability distribution of thresholds.

An apparently more pertinent criticism comes from Wilcox,[14] who found that under certain conditions visual acuity does not rise steadily with log I as previously found by every one, but that it becomes worse at high illuminations. In Wilcox's procedure two tiny, illuminated, vertical bars each subtending 2.4 by 20 *minutes* of visual angle are viewed against an absolutely black background, and the distance is determined by which the bars must be separated for them to be recognized as two bars. Thus the retina as a whole is completely dark, and only the very minute test objects are illuminated. It seems almost too elementary, but apparently quite necessary, to point out that the term retinal illumination refers to the general level of illumination of the retina as a whole, or of a goodly portion of it, and not of a minute part of it. What Wilcox measured is some aspect of glare, but not of visual acuity. That this analysis of Wilcox's difficulty is correct is shown when the reverse procedure is used, that is when the test bars are black and the background is evenly illuminated. Wilcox made such measurements also; they are in agreement with the classic data, and are therefore open to the same explanation.

6. *Comparisons.*—These criticisms thus leave untouched the explanation of the relation between visual acuity and illumination as due to a population distribution of the sensitive elements. This explanation, as well as the similar one concerned with intensity discrimination, has been taken over by Houstoun[6] with an enthusiasm which has failed to grasp the theoretical difficulties involved. For example, the original supposition was that visual acuity is a function of the *total* number of functional elements, but that intensity discrimination (though basically photochemical) is a function of the *differential increment* in the number of functional elements.

The present data on Drosophila and other data on the bee show that this idea is not tenable for intensity discrimination. If it were, then the intensity at which visual acuity alters most rapidly with log I should represent the most rapid rate of entrance of functional elements, and should therefore correspond to the place where intensity discrimination is best. Moreover, at high intensities when visual acuity practically ceases to increase because nearly all the elements are already functional and very few new ones enter, intensity discrimination should be poorest—in fact almost non-existent.

Yet neither of these things is true. Figure 4 shows that the greatest change of visual acuity with log I occurs at a point at which, as figure 2

shows, intensity discrimination is certainly not maximal. Moreover, at the highest intensities when visual acuity has reached its top value, intensity discrimination instead of being at its worst is actually at its best. Exactly the same is true for the bee, as shown by a similar comparison of the data of visual acuity with the recent data of Wolf on intensity discrimination. Very likely the same is true of the human eye, but the data are not all available at the moment.

It would thus seem that intensity discrimination is not dependent on the rate at which elements become functional, but rather on the total number of elements functional, as in visual acuity. For visual acuity this is simple, and is structurally sensible. For intensity discrimination it will call for a revision of ideas held up to now.

The full details of this work will be published in the *Journal of General Physiology*.

7. *Summary.*—The intensity discrimination and visual acuity of Drosophila may be measured at different illuminations by means of its reflex response to a moving visual pattern. Its visual acuity varies with log I as in man, the bee and the fiddler crab, being very poor at low illuminations and increasing in a sigmoid manner as log I increases. The maximum visual acuity is 0.0018, which is $^1/_{1000}$ of the maximum of the human eye and $^1/_{10}$ that of the bee.

The intensity discrimination of Drosophila is also poor. At low illuminations two lights are discriminated only when the ratio of their intensities is nearly 100. This ratio decreases as the intensity increases, and reaches a minimum of 2.5 which is maintained at the highest intensities. The minimum value of $\Delta I/I$ for Drosophila is 1.5, which is to be compared with 0.25 for the bee and 0.006 for man.

An explanation of the variation of visual acuity with illumination is given in terms of the variation in number of elements functional in the retinal mosaic at different intensities, this being dependent on the general statistical distribution of thresholds in the ommatidial population. Visual acuity is thus determined by the integral form of this distribution and corresponds to the total number of elements functional. The idea that intensity discrimination is determined by the differential form of this distribution is made untenable by the data, which indicate instead that intensity discrimination is probably also determined in some way by the total number of functional elements.

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[7] Hecht, S., *J. Gen. Physiol.*, **6**, 355 (1924).

[8] Aubert, H., *Physiologie der Netzhaut*, Breslau (1865).
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THE RELATIONSHIPS OF SCRAPTER, A GENUS OF AFRICAN BEES

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The introductory discussion is by the senior author, and the descriptive details which follow, as well as the figures, are by the junior author, but both writers have gone over the whole matter. The work has been done with the aid of a grant from the NATIONAL RESEARCH COUNCIL.

The details of Entomological taxonomy are usually considered of little interest except to specialists, but the insects do in fact throw much light on general biological problems. This has long been understood with reference to their cytology, and much has been made of their interesting and varied habits, but purely morphological details may also present aspects of general interest. The insects possess certain advantages for such studies, arising from the enormous numbers of genera and species, and the great complexity of many of their structures. Furthermore, they appear to represent, at least in very many cases, what may be called lateral evolution. That is to say, a type gives rise to other types, without itself becoming extinct, this being made possible by the different ecological situations, which permit the evolution of numerous non-competing species in the same general area. Furthermore, there is increasing proof that the genes for many characters lie latent for long periods, so that similar structures appear in different branches of the same family or order. It may of course be debated whether these reappearances are due to entirely new developments, or have a common origin somewhere in the ancestral germ plasm, or are due to a sort of orthogenesis. It is not necessary to assume that every case may be explained in the same way, but among the bees, at any rate, the evidence for germinal continuity and latency seems rather convincing.

Perhaps the most striking result of the recent studies of African bees has been the discovery that the genus *Hesperapis*, characteristic of the southwestern deserts of North America, is identical with *Capicola*, of the deserts of S. Africa. The genus has such excellent characters that it is not

credible that it developed independently in these two remote regions, any more than it is credible that the tsetse flies (*Glossina*) originated independently in Africa, where they now exist, and the Rocky Mountain Region, where they are found fossil.

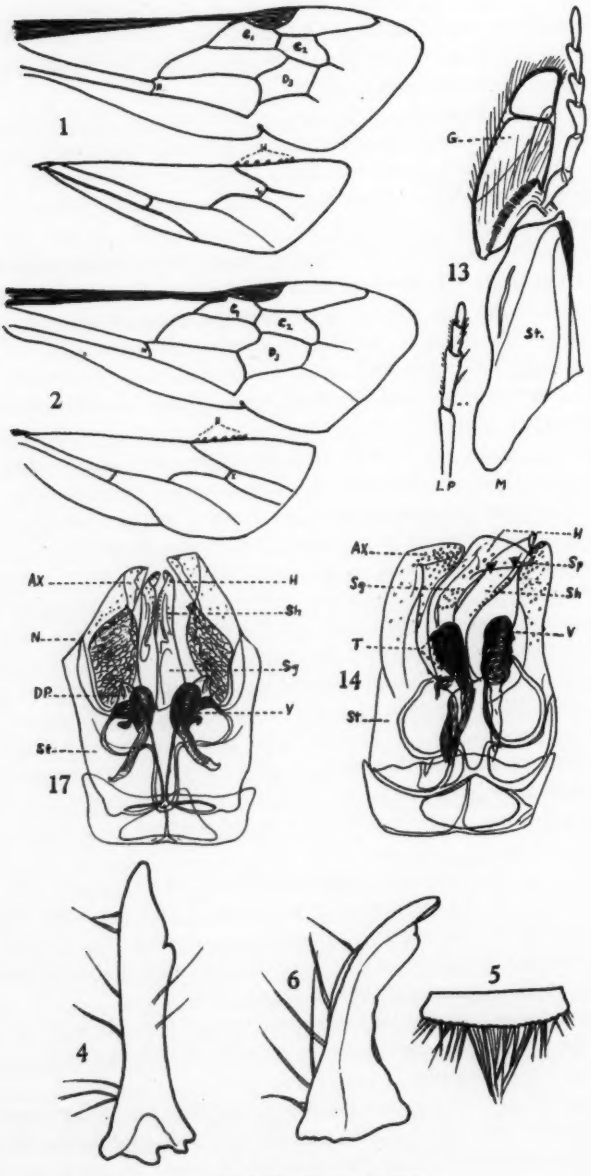
Among the most interesting groups of bees are the Diphaglossinae, related to *Colletes* of the Northern Hemisphere (which also extends southward in Africa and S. America) but forming a quite distinct subfamily. These Diphaglossinae, primitive bees with broad emarginate tongues, occur, with numerous genera, in Australia, S. America and S. Africa. They are, like the plants of the family Proteaceae, suggestive of ancient connections between these lands. In working over the African and Australian genera, it was noticed that the African genus *Scrapter*, Lepeletier and Serville, showed much resemblance to the Australian *Euryglossidia* Cockerell. In fact, on superficial examination they were not easy to separate. The question naturally arose, whether these genera were immediately related, or whether they had arisen from independent stocks, and represented convergent evolution. The more primitive forms of the subfamily have three cubital cells in the front wing; these genera have only two. It is, however, likely that the reduction in the number of cells has occurred independently, as it certainly has in several other groups of bees.

The morphological studies, as set forth below, appear to answer the question rather definitely in the sense of *Scrapter* and *Euryglossidia* having had no common ancestry with their special generic characters, though of course they both derive many characters from the common subfamily stem. To show more clearly the modifications of the characters used, as found in other bees, figures and notes have been added of the large Diphoglossine *Caupolicana;* of *Colletes*, type of the subfamily Colletinae; of *Palaeorhiza*, a peculiar Australian genus of the other short-tongued family, Hylaeidae; and finally of *Vespula*, which although exceedingly remote from the bees, nevertheless has a well-developed inner comb on the maxilla.

Comparison of Mouthparts.—In the mouthparts of the two bees *Scrapter semirufa* Ckll. and *Euryglossidia rectangulata* Ckll. the most striking difference is that in the shape of the labral processes, the narrow, rectangular plate and long hairs of *Euryglossidia* contrasting sharply with the squat, kite-shaped labrum and short hairs of *Scrapter*.

The mandibles of *Euryglossidia* are more curved than those of *Scrapter*, and have the inner tooth broad and somewhat double notched, whereas in *Scrapter* it is quite simple.

In the maxilla of *Scrapter*, an interesting point is the reduction of the number of teeth in the inner comb. In the specimen examined, one galea has two, one three, teeth, in the position a comb normally occupies. *Euryglossidia* has fourteen teeth on each galea, a smaller number than



Description on opposite page.

occurs in many bees, such as those figured. The wasp *Vespula* has a very long comb with a great number of teeth. In bees the teeth are usually a little longer at the lower end of the comb, but in *Euryglossidia* they are all about the same length, and in *Vespula* the lower teeth are even a little shorter than the upper ones.

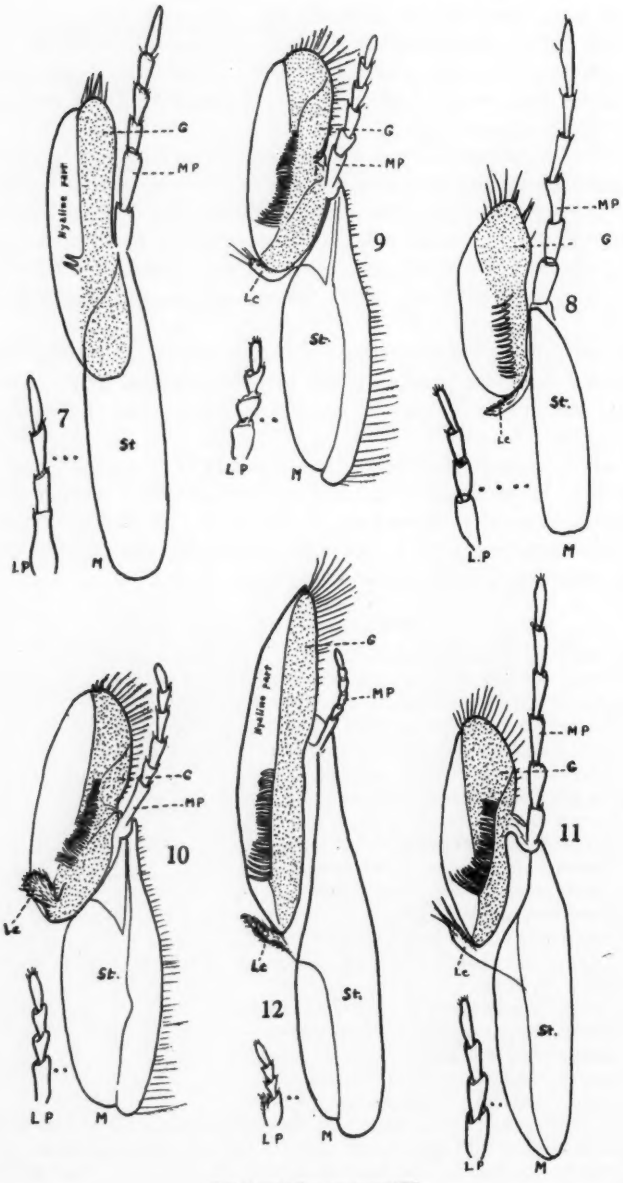
The maxillary palpi of *Euryglossidia*, composed of long, slender segments, are longer than the stipes, while the palpi of *Scrapter* are shorter than the stipes. Considerable variation occurs in the relation of these lengths, as, for example, the very small palp and large stipes of *Caupolicana yarrowi* Cresson, the short, thick palp and heavy stipes of *Colletes thoracicus* Smith, and the very long palp and rather delicate stipes of *Euryglossidia rectangulata*.

In *S. semirufa* the line separating the hyaline portion of the galea from the rest divides the galea nearly equally in a longitudinal line. In *E. rectangulata* the line is oblique. *Caupolicana yarrowi* has the comb lying along this line.

The whole galea is more slender and elongate in *S. semirufa* than in *E. rectangulata*. Contrast also the long, narrow galea of *C. yarrowi* with the bean-shaped ones of *C. thoracicus*, *C. inaequalis* Say and the Australian *Palaeorhiza parallela* (Ckll.). *Vespula* shows the transversely divided galea possessed also by some bees (Halictinae).

DESCRIPTION OF FIGURES

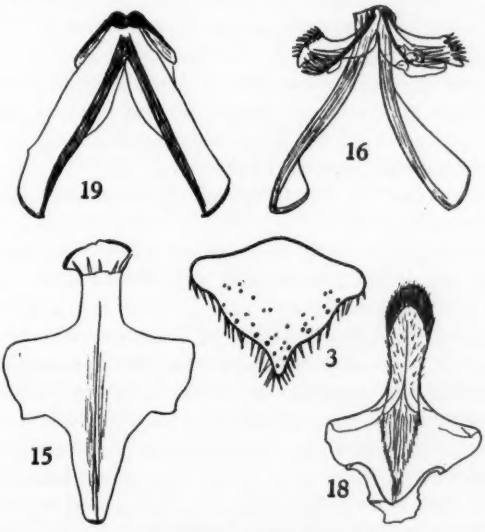
1. Scrapter semirufa, wings.
2. Euryglossidia rectangulata, wings. C_1–C_2, Cubital cells; D_3, Third discoidal cell; *H*, Hamuli; *I*, Intercubitella vein; *N*, Nervulus.
3. Scrapter semirufa, labrum.
4. Scrapter semirufa, mandible.
5. Euryglossidia rectangulata, labrum.
6. Euryglossidia rectangulata, mandible.
7. Scrapter semirufa, mouth parts.
8. Euryglossidia rectangulata, mouth parts.
9. Colletes thoracicus Sm., mouth parts.
10. Colletes inaequalis Say, mouth parts.
11. Palaeorhiza parallela Ckll., mouth parts.
12. Caupolicana yarrowi Cresson, mouth parts.
13. Vespula, mouth parts: *G*, Galea; *Lc*, Lacinia; *Lp*, Labial Palpus; *M*, Maxilla; *Mn*, Mentum; *St*, Stipes.
14. Euryglossidia rectangulata. Genital Armature.
15. Euryglossidia rectangulata. Inner Spatha.
16. Euryglossidia rectangulata. Outer Spatha.
17. Scrapter semirufa. Genital Armature.
18. Scrapter semirufa. Inner Spatha.
19. Scrapter semirufa. Outer Spatha. Abbreviations used for genital armature: *Ax*, Apical extension; *DP*, Dorsal projection of Stipes; *H*, Head of sagitta; *N*, Rudimentary notch; *Sg*, Sagitta; *Sh*, Shaft of Sagitta; *St*, Stipes; *T*, Teeth of Volsella; *V*, Volsella.



For description see page 975.

The labial palpi often give clues to classification. In *S. semirufa* the base of the first segment is enlarged, whereas in all the other bees figured the distal end is enlarged. Usually the whole first segment is stouter or longer than the others, as it is in *S. semirufa*, but seldom is it so much larger, relatively, as in *C. yarrowi*.

Unlike the bees illustrated, *Vespula* has hairs along the edge of the galea opposite the palp and practically none on the other edge. *Scrapter* has only a few hairs at the end of the galea, *Euryglossidia* a few more and *Colletes* and *Caupolicana* many, on the end and also along the edge toward the palp. Some of these hairs are plumose in *Colletes;* in the other bees they seem to be simple.



For description see page 975.

Comparison of Venation.—Though the wing venation of *S. semirufa* and *E. rectangulata* is very similar, a few differences can be found:

The first cubital cell of *Scrapter* is longer than the second, while in *Euryglossidia* the second is a little longer.

Correspondingly, the third discoidal cell is relatively wider at the top in *Euryglossidia* than in *Scrapter*.

Euryglossidia has a rather distinct vein from the third discoidal cell extending clear to the edge of the wing, while the vein in *Scrapter* is shorter and fainter.

The nervulus in *Euryglossidia* lies obliquely and does not curve uniformly, but that of *Scrapter* is almost an arc of a circle.

In the hind wing the two faint veins extending from the ends of the intercubitella toward the wing margin are short and diverge rather noticeably in *Scrapter*. In *Euryglossidia* they are longer and are nearly parallel.

Both bees have seven hamuli (hooks) on the edge of the hind wing which seem to be similarly grouped, the three most distal being closer together than the rest. And in spite of the differences of venation mentioned and a few other minor ones, the wings of the two bees are remarkably similar.

Comparison of Genitalia.—The genitalia of male bees are useful in differentiating species, but since they vary considerably among the species of a genus, they are of less value in separating genera. At a glance it may be seen, however, that the male genitalia of *Scrapter* and *Euryglossidia* are not at all alike, and some of the differences may be generic; though we cannot say definitely, having only one bee of each genus to study.

We notice that the sagittae of *E. rectangulata* are larger and longer than those of *S. semirufa* and possess each a short, stout spine on the head, while *S. semirufa* lacks them. Moreover, the head of the sagitta in *Euryglossidia* is larger than the shaft, whereas in *Scrapter* it is small and pointed and enlarges to meet the shaft.

No dorsal projections of the stipites are apparent in *Euryglossidia*, but *Scrapter* shows them plainly.

The stipites of *Scrapter* also may almost be called notched—that is, divided into an upper and a lower part by a sort of joint. *Euryglossidia* shows hardly any such tendency.

The volsellae of *Euryglossidia* have at least five well-defined teeth or notches along the outer edge; those of *Scrapter* are smoothly rounded. The basal parts of the volsellae are visible in *Scrapter* as two worm-like appendages, but they are very indistinct in *Euryglossidia*.

There are other differences in the morphology of the genital armatures which are evident from Figs. 14 and 17. The great dissimilarities in the shape of the outer and inner spathae (seventh and eighth abdominal sternites, respectively) are also quite evident from the drawings (Figs. 15, 16, 18, 19). Possibly the inner spatha of *Scrapter* serves the same purpose as sensory organ that perhaps the outer spatha of *Euryglossidia* does, for both are provided with hairs, while the other two spathae are almost or entirely nude. It seems probable that this is a generic difference. There is less chance that two species of one genus would be unlike in both of these two related parts to the extent of reversing their condition of pubescence, than that they should differ in only one part or in varying degree in both parts.

SPINORS IN PROJECTIVE RELATIVITY

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The extension to general relativity of the Dirac equations for the electron makes use of geometrical or physical objects with components which transform not only in response to changes of coördinates,

$$\bar{x}^i = \bar{x}^i(x) \tag{1.1}$$

($i = 1, 2, 3, 4$), but also in response to changes of gauge,

$$\bar{x}^0 = x^0 + \log \rho(x) \tag{1.2}$$

and to changes of spin-frame,

$$\bar{\varphi}_A = t_A^B \varphi_B \tag{1.3}$$

($A,B = 1, 2, 3, 4$). The components of certain geometrical objects, such as ordinary tensors, are completely determined when the coördinate system is fixed. Those of certain other objects, such as the electromagnetic potential or other projective tensors, have a degree of indetermination after the coördinates are fixed and become definite only when the gauge is fixed. The components of still other objects, the spinors, remain partially indeterminate after coördinates and gauge are fixed and become completely determinate only when the spin-frame is specified.

There are several ways of embodying this invariant theory in a formal calculus. The one which is here employed has its antecedents chiefly in the work of Weyl,[1] van der Waerden,[2] Fock[3] and Schouten.[4] It differs from the calculus arrived at by Schouten chiefly in the treatment of gauge invariance, Schouten having preferred (in collaboration with van Dantzig[5]) to rewrite the projective relativity in a formalism obtainable from the original one by a sort of coördinate transformation, whereas I think the original form fits in better with the classical notations of relativity theory. The treatment of the generalized Dirac equations from our point of view will be parallel in a general way to that outlined by Schroedinger[6] on physical and formal grounds, but will meet a number of questions more explicitly.

The theory of spinors is more general than the projective relativity and is reduced to the latter by the specification of certain fundamental spinors. These spinors have been recognized by several students[7] of the subject but their rôle has probably not been fully understood since it has quite recently been thought necessary to give special proofs of invariance. One of the fundamental spinors is the instrument in a scheme of raising and

lowering indices,[8] another one is the instrument for changing to and from dotted indices and a third serves to change projective tensor indices into pairs of spin indices.

Each step in the following discussion is determined by geometrical considerations,[9] but I have thought it desirable here to concentrate attention on the formal side and the paper may be read as a purely formal development of equations with a physical application in view.

2. The variables $x^1, \ldots, x^4$ are real and are the coördinates of space-time. The four arbitrary functions which define a transformation of coordinates (1.1) are real. So are the gauge variable and the one arbitrary function which enters in a gauge transformation. The sixteen arbitrary functions $t_B^{\dot{1}}$ which enter in a spin-transformation (1.3) are complex functions of x^1, x^2, x^3, x^4 subject only to the condition that the determinant,

$$\left| t_B^{\dot{A}} \right| = t \neq 0. \tag{2.1}$$

The components of spinors are in general complex numbers.

The gauge transformations are completely independent of the coördinate transformations, and the spin transformations are completely independent of both coördinate and gauge transformations.

As an example of a spinor we may consider one which has four components of the form

$$\psi^A = e^{Nx^0} f^A(x^1,x^2,x^3,x^4)$$

and a law of transformation,

$$\bar{\psi}^A = \psi^B T_B^{\dot{1}} t^k$$

where T_B^A are defined by the relation

$$T_B^{\dot{1}} t_C^B = \delta_C^A. \tag{2.2}$$

Under coördinate and gauge transformations the components behave as components of four scalars. The number N is called the *index* of the spinor. In a typical case $N = \sqrt{-1}$ and we will also have cases in which $N = 0$. The number k is called the *weight* and in a typical case we have $k = 1/4$.

Capital Latin letters $A, B, \ldots$, etc., as subscripts or superscripts take on the values 1, 2, 3, 4. Small Greek letters $\alpha, \beta, \ldots$ from the first part of the alphabet will take on the values 0, 1, 2, 3, 4.

As a more complicated example consider a spinor which has 80 components of the form

$$\gamma_B^{A\alpha} = f_B^{\dot{1}}{}_\alpha(x^1,x^2,x^3,x^4) \tag{2.3}$$

and the law of transformation,

$$\gamma_B^{A\alpha} = \gamma_D^{C\beta} \frac{\partial \bar{x}^\alpha}{\partial x^\beta} T_C^A t_B^{D*} t^{\dot{1}} t^{-\dot{1}*}. \tag{2.4}$$

The star * indicates the complex conjugate of the quantity to which it is applied. A dot[10] over a spin index (B and D in (2.4)) indicates that the transformation coefficients which affect it are the complex conjugates of T_B^A or t_B^A as the case may be.

In this case the index of the spinor is zero since the components are functions of $x^1, \ldots, x^4$ alone. The spinor has two weights, a direct weight of $^1/_4$ corresponding to the non-dotted spin index, and a conjugate weight of $-^1/_4$ corresponding to the dotted one.

3. In the general spinor analysis pairs of alternating indices may be raised and lowered by means of the special alternating spinors $^1/_{2\epsilon}{}^{ABCD}$ and $^1/_{2\epsilon_{ABCD}}$ which take on the values 0, 1, -1 according to the familiar rule and are of weight $+1$ and -1, respectively. Thus if

$$X^{AB} = -X^{BA} \tag{3.1}$$

we write

$$X_{AB} = {}^1/_{2\epsilon_{ABCD}} X^{CD}. \tag{3.2}$$

The antisymmetric spinors X^{AD} of weight $^1/_2$ which satisfy the condition

$$X_{AB} X^{AB} = {}^1/_{2\epsilon_{ABCD}} X^{AB} X^{CD} = 0 \tag{3.3}$$

are called *special*. Those for which this condition is not satisfied are called general. The special ones are capable of being represented in the form

$$X^{AB} = {}^1/_2(\psi^A \varphi^B - \psi^B \varphi^A) \tag{3.4}$$

where ψ^A and φ^A are spinors of weight $^1/_4$ each.

In the geometric interpretation in which ψ^A are regarded as homogeneous coördinates in a complex projective 3-space P_3, the special X^{AB}'s are Pluecker coördinates of lines and the general X^{AB}'s represent null systems (or the linear complexes of these null systems).

4. In the spinor analysis of projective relativity there is a set of three invariant or fundamental spinors γ_{AB}, γ_B^A and $\gamma_{A\dot{B}}$, any two of which determine the third. The spinor γ_{AB} represents a general null system in the 3-space P_3. It satisfies the condition,

$$\gamma_{AB} = -\gamma_{BA} \tag{4.1}$$

and is normalized so that

$$\gamma_{AB}\gamma^{AB} = -4 \tag{4.2}$$

from which it follows when we take account of the fact that A and B run from 1 to 4 that

$$\gamma_{AB}\gamma^{BC} = \delta_A^C. \tag{4.3}$$

The spinor $\gamma_{\dot{B}}^{A}$ represents an antiinvolution of the second kind. It is normalized so that

$$\gamma_{\dot{B}}^{A}\gamma_{C}^{\dot{B}} = \delta_{C}^{A} \tag{4.4}$$

where

$$\gamma_{C}^{\dot{B}} = -\gamma_{\dot{C}}^{B^*}. \tag{4.5}$$

The antiinvolution and null system must be commutative geometric transformations, i.e., each must transform the other into itself. This means

$$\gamma_{\dot{C}}^{A}\gamma_{\dot{D}}^{B}\gamma_{AB} = \overset{*}{\gamma}_{CD} \tag{4.6}$$

or

$$\gamma_{\dot{C}}^{A}\gamma_{BA} = (\gamma_{CD}\gamma_{\dot{B}}^{D})^*. \tag{4.7}$$

Thus if we define

$$\gamma_{B\dot{C}} = \gamma_{BA}\gamma_{\dot{C}}^{A} \tag{4.8}$$

we have

$$\gamma_{B\dot{C}} = \overset{*}{\gamma}_{C\dot{B}} \tag{4.9}$$

so that $\gamma_{B\dot{C}}$ are the elements of a Hermitian matrix. This spinor represents a hyperbolic antipolarity of the second kind. The points which lie on their antipolar planes lie on an antiellipsoid whose equation is

$$\gamma_{B\dot{C}}\psi^{B}\psi^{C^*} = 0.$$

This Hermitian form is of signature $(+\ +\ -\ -)$. The same antipolarity is represented by the spinors $\gamma_{\dot{C}B}$ and $\gamma^{\dot{C}B}$ and $\gamma^{B\dot{C}}$, where

$$\gamma_{\dot{C}B} = -\gamma_{B\dot{C}} \qquad \gamma^{\dot{B}C} = -\gamma^{C\dot{B}}$$
$$\gamma^{B\dot{C}} = \gamma^{BA}\gamma_{A}^{\dot{C}} \qquad \gamma_{\dot{C}A}\gamma^{B\dot{C}} = \delta_{A}^{B}.$$

The spinor γ_{AB} is of weight $-1/2$, $\gamma_{\dot{B}}^{A}$ is of direct weight $+1/4$ and conjugate weight $-1/4$, $\gamma_{A\dot{B}}$ is of direct weight $+1/4$ and conjugate weight $-1/4$. The three geometric transformations represented by γ_{AB}, $\gamma_{\dot{B}}^{A}$ and $\gamma_{A\dot{B}}$ are commutative with one another and, together with the identity, constitute a 4-group.

5. The three fundamental spinors give a scheme for raising and lowering spin indices and also for changing from dotted to undotted indices and back again. Thus

$$\psi_{A} = \gamma_{AB}\psi^{B}$$
$$\psi^{A} = \gamma^{AB}\psi_{B}$$
$$\psi_{\dot{A}} = \gamma_{\dot{A}B}\psi^{B} = \gamma_{\dot{A}}^{B}\psi_{B}$$
$$\psi^{\dot{A}} = \gamma^{\dot{A}B}\psi_{B}$$

and so on. In raising and lowering indices we always sum with respect to the *second* index of γ^{AB}, $\gamma^{A\dot{B}}$, $\gamma_{A\dot{B}}$, $\gamma_{\dot{A}B}$, etc.

It is well to remember that $\varphi_{A}\psi^{A} = -\varphi^{A}\psi_{A}$.

In order to be unambiguous in the process of raising and lowering indices it is necessary to keep track of the order of the indices both above and below. This may be done, following Schouten, by the use of dots to fill the places from which indices have been raised or lowered. But we shall be able to get along for the present with the special convention that when the order of an upper and lower index is not otherwise indicated *the upper index comes first.*

The conventions for the raising and lowering of indices by means of the γ spinors are in agreement with the one given for pairs of alternating indices in §3 above in the case in which

$$\gamma_{AB}X^{\cdots AB\cdots} = 0$$

and in this case only. For the equations

$$\gamma_{AB}X^{AB} = 0 \text{ and } X^{AB} = -X^{BA}$$

imply, since A and B run from 1 to 4, that

$$\gamma_{AB}X^{CB} = -X_{AB}\gamma^{CB} \tag{5.1}$$

where

$$X_{AB} = {}^1/_{2}\epsilon_{ABCD}X^{CD},$$

and (5.1) implies

$$\gamma_{AC}\gamma_{BD}X^{CD} = X_{AB}.$$

Thus, for example, we cannot apply the rules for changing individual indices to the γ spinors themselves. The chief application of the full set of rules is to the class of spinors considered in the next section.

6. Consider the set of antisymmetric spinors X^{AB} which satisfy the conditions

$$\gamma_{AB}X^{AB} = 0 \tag{6.1}$$

$$\gamma^{A}_{\dot{C}}\gamma^{B}_{\dot{D}}X^{CD*} = X^{AB}. \tag{6.2}$$

Each such spinor has sets of components,

$$X_{AB},\ X^{A}_{B},\ X^{\dot{A}}_{B},\ X^{A}_{\dot{B}},\ X^{\dot{A}\dot{B}},\ X_{A\dot{B}},\ X_{\dot{A}\dot{B}}.$$

In particular

$$X^{A}_{B} = \gamma^{AC}X_{CB} = \gamma_{BC}X^{AC}. \tag{6.3}$$

$$-X^{A}_{B} = \gamma^{AC}X_{BC} = \gamma_{BC}X^{CA}. \tag{6.4}$$

$$X^{\dot{A}\dot{B}} = X^{AB*}. \tag{6.5}$$

We relegate a few remarks about the geometric meaning of these spinors to a footnote.[11] Since the indices run exactly from 1 to 4, we have

$$X^{A}_{B}X^{B}_{C} = {}^1/_{4}\ X\ \delta^{A}_{C} \tag{6.6}$$

where

$$X = X^{AB}X_{AB}. \tag{6.7}$$

If L and M are two of the spinors X^A_B, the product

$$P^A_B = L^A_C M^C_B$$

consists of terms of which

$$P^1_1 = L^1_B M^B_1 = (L^{12}M^{34} + L^{13}M^{42} + L^{14}M^{23})$$

and

$$P^1_2 = L^1_B M^B_2 = L^{13}M^{14} - L^{14}M^{13}$$

are typical. Hence

$$L^A_B M^B_C + M^A_B L^B_C = {}^1/_2 P\, \delta^A_C \tag{6.8}$$

where

$$P = L^A_B M^B_A = L^{AB}M_{AB}. \tag{6.9}$$

Also the spinor,

$$L^A_B M^B_C - M^A_B L^B_C = S^A_C \tag{6.10}$$

satisfies the conditions,

$$S^A_A = 0, \tag{6.11}$$

$$S_{AC} = \gamma^{BD}(L_{AB}M_{DC} - M_{AB}L_{DC}) = S_{CA}, \tag{6.12}$$

and

$$\begin{aligned} S^*_{A\dot{C}} &= (L_{AB}M^B_{\dot{C}} - M_{AB}L^B_{\dot{C}})^* \\ &= -\gamma^P_A \gamma^Q_{\dot{B}}(L_{PQ}M^{\dot{B}}_C - M_{PQ}L^{\dot{B}}_C) \\ &= L^Q_C M_{\dot{A}Q} - M^Q_C L_{\dot{A}Q} \\ &= S_{C\dot{A}}. \end{aligned} \tag{6.13}$$

Furthermore the matrix S anticommutes with L and M, i.e.,

$$L^A_B S^B_C = -S^A_B L^B_C. \tag{6.14}$$

The spinor S^A_B represents a line reflection whose directrices are lines of the linear complex γ_{AB} and S_{AB} is a polarity with respect to a quadric having one system of generators in the linear complex γ_{AB}.

7. The totality of skew symmetric matrices X^{AB} is a six parameter linear family, that is, expressible in the form,

$$X^{AB} = X^\sigma K^{AB}_\sigma \tag{7.1}$$

where σ runs from 0 to 5, and K^{AB}_σ are six linearly independent skew-symmetric matrices. If the X^σ's are interpreted as homogeneous coördinates in a complex 5-space P_5, these equations represent a (1–1) correspondence

between the points of P_5 and the null systems of P_3. The special null systems, i.e., the lines of P_3, correspond to the points of a quadric

$$\gamma_{\sigma\tau}X^\sigma X^\tau = 0 \tag{7.2}$$

in which (cf. 3.3)

$$\gamma_{\sigma\tau} = {}^1/_{2\epsilon_{ABCD}} K_\sigma^{AB} K_\tau^{CD},$$

σ and τ running from 0 to 5.

Since the correspondence represented by the equations (7.1) is (1–1) they have an inverse

$$X^\sigma = K_{AB}^\sigma X^{AB} \tag{7.3}$$

where

$$K_\sigma^{AB} K_{AB}^\tau = \delta_\sigma^\tau \tag{7.4}$$

and

$$K_\sigma^{AB} K_{CD}^\sigma = {}^1/_2 \delta_{CD}^{AB}. \tag{7.5}$$

Since, $K_{AB}^0, \ldots, K_{AB}^5$ can be any six linearly independent null systems we may let

$$K_{AB}^5 = {}^1/_2 \gamma_{AB} \tag{7.6}$$

and let the other five be such that they all satisfy the conditions (6.1) and (6.2). Moreover the coördinates in P_3 may be chosen so that γ_{AB} and γ_B^A have the matrice of components

$$\begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} \tag{7.7}$$

respectively. In this coördinate system a special choice of the transformation (7.3) satisfying (7.6) is

$$\begin{aligned} X^0 &= X^{12} - X^{34} & X^5 &= i(X^{12} + X^{34}) \\ X^2 &= X^{14} - X^{23} & X^1 &= i(X^{14} + X^{23}) \\ X^4 &= i(X^{13} - X^{42}) & X^3 &= i(X^{13} + X^{42}). \end{aligned} \tag{7.8}$$

Under this form of the transformation the quadric (7.2) which is the image of (3.3) becomes

$$-(X^0)^2 - (X^1)^2 - (X^2)^2 - (X^3)^2 + (X^4)^2 - (X^5)^2 = 0. \tag{7.9}$$

The null systems X^{AB} which are commutative with $\gamma_{\alpha\beta}$, i.e., satisfy (6.1), correspond to the points on the hyperplane

$$X^5 = 0 \tag{7.10}$$

in P_5 and those which are commutative with γ_B^A (i.e., satisfy (6.2)) correspond to points on the locus,

$$X^\sigma = \text{real.} \tag{7.11}$$

Thus the null systems of P_3 which satisfy (6.1) and (6.2) correspond to the points of a real four-dimensional space T_4, and the special null systems (straight lines) correspond to the points on a quadric which in a special coordinate system has the equation,

$$-(X^0)^2 - (X^1)^2 - (X^2)^2 - (X^3)^2 + (X^4)^2 = 0. \tag{7.12}$$

8. The space T_4 we identify with the tangent space, and the quadric (7.12) with the quadric[12]

$$\gamma_{\alpha\beta}X^\alpha X^\beta = 0 \tag{8.1}$$

(α and β now run from 0 to 4) of the projective relativity. There is known to be a linear transformation, with real coefficients, A^α_β, in the tangent space T_4 which carries a coördinate system in which the quadric has the equation (7.12) into one in which it has the equation (8.1). Let

$$\gamma^\alpha_{AB} = A^\alpha_\beta K^\beta_{AB}. \tag{8.2}$$

Then

$$X^\alpha = \gamma^\alpha_{AB}X^{AB} \tag{8.3}$$

is a transformation which carries the spinors satisfying (6.1) and (6.2) into the points of the tangent space T_4 and the subset of special spinors (3.3) into the points of the projective relativity quadric (8.1). The inverse of (8.3)

$$X^{AB} = \gamma^{AB}_\alpha X^\alpha$$

is obtained from (7.1) by a formula analogous to (8.2), and (7.4) and (7.5) translate into

$$\gamma^{AB}_\alpha \gamma^\beta_{AB} = \delta^\beta_\alpha \tag{8.4}$$

and

$$\gamma^{AB}_\alpha \gamma^\alpha_{CD} = {}^1/_2\,\delta^{AB}_{CD} + {}^1/_4\,\gamma^{AB}\gamma_{CD}, \tag{8.5}$$

respectively. The relation between the projective relativity quadric and the quadric (3.3) is given by

$$\gamma_{\alpha\beta} = {}^1/_2\,\epsilon_{ABCD}\gamma^{AB}_\alpha \gamma^{CD}_\beta. \tag{8.6}$$

This equation is invariant if γ^{AB}_α is a spinor of weight $+\,{}^1/_2$ of the type indicated by its indices.

9. Greek indices are, of course, raised and lowered by means of the projective tensors $\gamma^{\alpha\beta}$ and $\gamma_{\alpha\beta}$, and this convention is consistent with the relations established above, such as (8.4) and (8.5).

The three spinors γ_{AB}, γ^A_B and γ^α_{AB} embody the complete system of relations between the antisymmetric spinors X^{AB} and the tangent spaces of space-time. The first spinor cuts the dimensionality of the set from five

to four complex dimensions, the second singles out a real subspace and the third defines the correspondence explicitly.

Under our conventions as to indices the correspondence can be put in various forms which represent the tangent space T_4 upon various families of objects in P_3. Examples are:

$$X^A_B = \gamma^A_{B\alpha} X^\alpha, \quad \text{or} \quad X = \gamma_\alpha X^\alpha, \tag{9.1}$$

if we denote the matrix X^A_B by X and the matrix $\gamma^A_{B\alpha}$ by γ_α;

$$X^A_{\dot B} = \gamma^A_{\dot B\alpha} X^\alpha, \tag{9.2}$$

$$X^\alpha = \gamma^\alpha_{A\dot B} X^{A\dot B}. \tag{9.3}$$

For the points on the projective relativity quadric, we have

$$X^\alpha = \gamma^\alpha_{A\dot B} \psi^A \psi^{B*} \quad \text{and} \quad \gamma_{A\dot B} \psi^A \psi^{\dot B} = 0. \tag{9.4}$$

This is a correspondence between the points of an anti-quadric in the spin-space P_3 and the points of the quadric such that all points on any invariant line of γ^A_B correspond to the same point of the quadric.

10. The transformation (9.1) is particularly interesting. It represents the points of the tangent space T_4 upon line reflections such as we studied in §6 and §7. Hence it represents T_4 by a system of pairs of lines in P_3. For each value of α, the matrix γ^α itself represents a line reflection which transforms γ_{AB} and $\gamma^A_{\dot B}$ into themselves. The relation (6.8) between any two of these line reflections translates into

$$\gamma^\alpha \gamma^\beta + \gamma^\beta \gamma^\alpha = {}^1/_2 \gamma^{\alpha\beta} 1. \tag{10.1}$$

The theorems about the spinor (6.10) imply that if

$$s^{\alpha\beta} = {}^1/_2 (\gamma^\alpha \gamma^\beta - \gamma^\beta \gamma^\alpha) \tag{10.2}$$

$s^{\alpha\beta}$ is a line reflection. This means

$$s^{A\alpha\beta}_A = 0, \; s^{\alpha\beta}_{AB} = s^{\alpha\beta}_{BA}, \; s^{\alpha\beta}_{A\dot B} = s^{\alpha\beta}_{B\dot A}{}^*. \tag{10.3}$$

The relation (8.5) goes into

$$\gamma^A_{B\alpha} \gamma^{C\alpha}_D + {}^1/_4 \delta^A_B \delta^C_D = {}^1/_2 \delta^{AP}_{QD} \gamma_{BP} \gamma^{CQ}. \tag{10.4}$$

But because the indices run from 1 to 4

$$\delta^{AP}_{QD} \gamma_{BP} \gamma^{CQ} = \delta^C_B \delta^A_D + \gamma_{BD} \gamma^{CA}. \tag{10.5}$$

Hence

$$\gamma^A_{B\alpha} \gamma^{C\alpha}_D = -{}^1/_4 \delta^A_B \delta^C_D + {}^1/_2 \delta^C_B \delta^A_D + {}^1/_2 \gamma_{BD} \gamma^{CA}. \tag{10.6}$$

Hence

$$\gamma^A_{B\alpha} \gamma^{B\alpha}_D = {}^5/_4 \delta^A_D. \tag{10.7}$$

This agrees with the formula,

$$\gamma_\alpha\gamma^\beta + \gamma^\beta\gamma_\alpha = {}^1/_2\,\delta^\beta_\alpha\,1 \qquad (10.8)$$

which is another form of (10.1).

11. The spinors obtainable from γ^α_{AB} by juggling indices can be used to convert projective indices into spinor indices. Thus, for example, in the projective relativity the contact point of the tangent space is represented by

$$\varphi^\alpha = \delta^\alpha_0$$

and this has covariant components φ_α which have been interpreted as components of electromagnetic potential. We thus find

$$\varphi^A_B = \gamma^A_{B\alpha}\varphi^\alpha = \gamma^A_{B0}.$$

This is in agreement with the fact that the null component of a projective vector is a scalar.

The process of changing projective indices into spin indices enables us to translate projective differentiation of spinors into a process of covariant differentiation of spinors entirely in terms of spin indices.

[1] H. Weyl, "Gravitation and the Electron," these PROCEEDINGS, **15**, 323 (1929); "Elektron und Gravitation," *Zeitsch. Phys.*, **56**, 330 (1929).

[2] B. L. van der Waerden, "Spinoranalyse," *Goettinger Nachrichten*, 100 (1929). This spinor analysis was expounded and developed by O. Laporte and G. E. Uhlenbeck, "Application of Spinor Analysis to the Maxwell and Dirac Equations," *Phys. Rev.*, **37**, 1380 (1931). A two-component spinor analysis for general relativity is given by L. Infeld and B. L. van der Waerden, "Die Wellengleichung des Elektrons in der allgemeinen Relativitätstheorie," *Sitzungsberichte der Preussische Akademie der Wissenschaften*, **9**, 380 (1933).

[3] V. Fock, "Geometrisierung der Diracschen Theorie des Elektrons," *Zeitsch. Phys.*, **57**, 261 (1929), and V. Fock, "L'équation d'onde de Dirac et la géométrie de Riemann," *Journal de Physique et le Radium*, Serie VI, Tome X, No. 11, 392–405 (1929).

[4] J. A. Schouten, "Dirac Equations in General Relativity. 1. Four Dimensional Theory. 2. Five Dimensional Theory," *Journ. Math. and Phys.*, **10**, 239 (1931).

[5] J. A. Schouten and D. van Dantzig, "Zur generellen Feldtheori," *Zeitsch. Phys.*, **78**, 639 (1932); Schouten, "Generelle Feldtheorie, Ableitung des Impulsenergiestromprojektore aus einem Variationsprinzip," *Ibid.*, **81**, 129 (1933); Schouten, "Zur generellen Feldtheorie, Raumzeit und Spinraum," *Ibid.*, **81**, 405 (1933); Schouten and van Dantzig, "On Projective Connections and Their Application to the General Field-Theory," *Ann. Math.*, **34**, 271 (1933); Schouten, "Zur generellen Feldtheorie: Semivektoren und Spinraum," *Zeitsch. Phys.*, **84**, 92 (1933). This last paper comments on one by A. Einstein and W. Mayer, "Semivektoren und Spinoren," *Sitzungsberichte der Preussischen Akademie der Wissenschaften*, 37 (1932).

[6] E. Schroedinger, "Diracsches Elektron im Schwerefeld 1," *Sitzungsberichte der Preussischen Akademie der Wissenschaften*, Phys.-Math. Klasse 11 (1932), and "L'electron de Dirac dans la theorie de la rélativité générale," *Congres International d'electricité*, Paris (1932).

[7] W. Pauli and J. Solomon, "La theorie unitaire d'Einstein et Mayer et les èquations de Dirac, I and II," *Journ. de Phys. et le Rad.*, **3** (series 7), 452, 582 (1932). See also

Note 5. Fock (Note 3) had the invariant Hermitian form $\gamma_{A\dot{B}}\psi^A\psi^{B*}$ and, since he was dealing with general relativity, also treated $\gamma^0_{A\dot{B}}\psi^A\psi^{B*}$ as an invariant.

[8] The spinors with which we raise and lower indices are not analogous to those used in the two-component theory by van der Waerden and Infeld (see Note 2). The analogue of their process is the raising and lowering of pairs of alternating indices in §3.

[9] This geometry centers around the Pluecker-Klein correspondence between the straight lines of a 3-space and the points of a quadric in a 5-space. Although I have been speaking and lecturing about this since early in 1930, I neglected to publish anything until this year (this volume of these PROCEEDINGS, p. 503), and although I communicated the idea to Schouten when he was lecturing on spinors in Princeton in the winter of 1930–31, he has not used it until his paper in *Zeitsch. Phys.*, **81** (cf. Note 5). It also appears in K. Nikolsky, "Zur Theorie der Spinoren," *Zeitsch. Phys.*, **83**, 284 (1933).

[10] This convention comes from B. L. van der Waerden, "Spinoranalysis," *Göttingen Nachrichten*, 100 (1929). The theory of certain geometric objects with complex conjugates in the law of transformation has been studied by Schouten and van Dantzig, "Ueber unitäre Geometrie," *Math. Ann.*, **103**, 319 (1930).

[11] In case X_{AB} is special it represents a line which is both in the linear complex represented by γ_{AB}, and in the anticongruence of invariant lines of the antiinvolution γ^A_B and also on the antiquadric $\gamma_{A\dot{B}}$. In case X_{AB} is general the null system represented by it is commutative with the null system represented by γ_{AB}. Algebraically this corresponds to the fact embodied in equations (6.3) and (6.4) that X_{AB} and γ_{AB} are anticommutative. Each of the null systems γ_{AB} and X_{AB} transforms the other into itself. They are said to be involutoric. They could also be called orthogonal because they are conjugate with respect to the quadric (3.3).

When X_{AB} is general X^A_B and $-X^A_B$ represent the collineation which is the product in either order of X_{AB} and γ_{AB}. This collineation is of period two because of (6.6). It leaves invariant the lines of the linear congruence common to the linear complexes γ_{AB} and X_{AB} and therefore leaves the two directrices l_1, l_2 of this congruence pointwise invariant. The collineation is thus a line reflection; i.e., it transforms each point ψ of P_3 into the point $\bar{\psi}$ such that the line $\psi\bar{\psi}$ meets l_1 and l_2 in points which are separated harmonically by ψ and $\bar{\psi}$.

[12] At the present stage we make no assumption as to the index of the projective tensor $\gamma_{\alpha\beta}$. The index of $\gamma^\alpha_{A\dot{B}}$ is, of course, determined by that of $\gamma_{\alpha\beta}$.

NOTE OF CORRECTIONS IN PAPERS BY W. A. ZISMAN, THESE PROCEEDINGS, JULY NUMBER, 1933

Due to a regrettable error in calculation of Poisson's ratio (statistical values) all values are 5.5 per cent too small. This correction should be applied to the values of the ratio given in table 3, page 659; table 4, page 660; on page 681 (Quincy granite); page 682 (Sudbury norite); page 683, first paragraph (limestone). Correspondingly some of the calculated values of compressibilities or rocks, given in table 1, page 683, should be slightly increased. These errors do not affect the correctness of the conclusion derived from the data of the table and stated in the text.

In all cases the values of Poisson's ratio and of the compressibilities described is correctly stated in the separates of Dr. Zisman's papers. Such corrected separates will be procurable by applying to the Secretary of the Division of Geological Sciences, Geological Museum, Oxford St., Cambridge, Massachusetts.

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